MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

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1493. If a conic have double contact with two fixed conics, find the envelope of its chords of intersection with another fixed conic having double contact with one of the two fixed conics. 40

1497. (1.) Given three points by equations of the form $lx+my+nz = 0$; prove that the area of the triangle contained by them is

$$(l_1 m_2 n_3) \div (l_1+m_1+n_1)\ (l_2+m_2+n_2)\ (l_3+m_3+_3),$$

that of the triangle of reference being unity.
(2.) Also, if (123) denote the area of the triangle contained by the points 1, 2, 3, and so on, prove that

$$(123)\ (456) \equiv (156)\ (423) + (164)\ (523) + (145)\ (623).$$ ... 53

1551. Using the notation of Spence's Transcendents, viz.,

$$L^n(1+x) = \frac{x}{1^n} - \frac{x^2}{2^n} + \frac{x^3}{3^n} - \frac{x^4}{4^n} + \text{\&c., prove that}$$

$$L^2(1+x) + L^2\left(\frac{1}{1+x}\right) = \tfrac{1}{2}\log^2(1+x); \text{ also that}$$

$$L^2\left(\frac{1}{1+x}\right) + L^2\left(\frac{x}{1+x}\right) = \log^2(1+x) - \log x \log(1+x) - \tfrac{1}{6}\pi^2.$$ 96

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1672. Give simple geometrical constructions for (1) inscribing the greatest triangle in an ellipse, and (2) circumscribing the least triangle about an ellipse. 91

1675. If a triangle abc be the reciprocal of ABC in respect of a parabola whose parameter is $4m$; and if n_1, n_2, n_3 be the normals at the vertices of diameters through ABC; then

$$\frac{(\text{area of } abc)^2}{bc\,.\,ca\,.\,ab} = \frac{2m^2}{n_1 n_2 n_3}\,(\text{area of ABC}).$$ 90

1677. Prove that a chord of constant length, drawn through either point of contact of a circle which has double contact with a fixed conic, envelopes a confocal conic. 42

1684. Prove by inversion the leading properties of the Lemniscate of Bernoulli, including those of its osculating circle and Saladini's property that it is the curve down which a particle will descend through any arc in the same time as it will take to descend through the chord of the arc. Show also that it possesses, in a modified way, anharmonic properties similar to those of the conic sections. 47

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1692. Prove that, n being any positive integer,

$$\frac{2^{2n}}{\underline{|2n+1}} = \frac{1}{\underline{|2n+1}} + \frac{1}{\underline{|2n-1}}\frac{1}{\underline{|2}} + \frac{1}{\underline{|2n-3}}\frac{1}{\underline{|4}} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{\underline{|2n}},$$

$$\frac{2^{4n-2}}{\underline{|4n}} = \frac{1}{2}\frac{1}{(\underline{|2n})^2} + \frac{1}{\underline{|2n-2}}\frac{1}{\underline{|2n+2}} + \frac{1}{\underline{|2n-4}}\frac{1}{\underline{|2n+4}} + \ldots\ldots + \frac{1}{\underline{|4n}},$$

$$= \frac{1}{\underline{|2n-1}}\frac{1}{\underline{|2n+1}} + \frac{1}{\underline{|2n-3}}\frac{1}{\underline{|2n+3}} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{\underline{|4n-1}},$$

$$\frac{2^{4n}}{\underline{|4n+2}} = \frac{1}{2}\frac{1}{(\underline{|2n+1})^2} + \frac{1}{\underline{|2n-1}}\frac{1}{\underline{|2n+3}} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{\underline{|4n+1}},$$

$$= \frac{1}{\underline{|2n}}\frac{1}{\underline{|2n+2}} + \frac{1}{\underline{|2n-2}}\frac{1}{\underline{|2n+4}} + \frac{1}{\underline{|2n-4}}\frac{1}{\underline{|2n+6}} + \ldots + \frac{1}{\underline{|4n+2}}$$ 62

1695. If u_m be a homogeneous function of the degree m in any number of independent variables, prove that

$$\left(x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz}\ldots\right)\log u_m, \text{ identically};$$

$$\left(x^2\frac{d^2}{dx^2} + y^2\frac{d^2}{dy^2}\,\&c. + 2xy\frac{d}{dx}\cdot\frac{d}{dy}\,\&c.\right)\log u_m = -m;$$

and continue the theorem. 39

1696. Supplemental chords of an ellipse are produced to meet the tangents at the extremities of the corresponding diameter: prove that the rectangle contained by the intercepts on the tangents is constant. 43

1697. If two circles touch a straight line a, and have a different kind of contact with two other straight lines b and c, prove that the middle point of the distance of a from the point of intersection of b and c lies in a straight line with the centre of one of the two circles and the point where the other circle touches a.

N.B.—The contact of two circles with two straight lines is said to be like, or of the same kind, when the two circles are on the *same side* of *both* lines, or else on *different sides* of *both* lines; but if the two circles lie on the same side of one of the two straight lines, and on different sides of the other straight line, then the contact of the two circles with the two straight lines is said to be unlike, or of different kinds. 98

1698. Prove that the foot of a perpendicular on a common tangent (a) to two circles, drawn from the point of intersection of the two contrary common tangents (b and c) to those circles, lies in a straight line with the centre of one of the circles and the reflexion of the centre of the other with respect to the tangent (a). 98

1704. From any point in a straight line perpendiculars are drawn on the sides of a triangle, and the feet of these perpendiculars joined; find the locus of the centre of gravity (or *centroid*) of the triangle thus formed. 60

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1707. If A, B, C are the distances of a line from the angles of a triangle which it cuts in the points d, e, f respectively; then $$\frac{A\,(B-C)}{ef}=\frac{B\,(C-A)}{fd}=\frac{C\,(A-B)}{de}=\frac{(B-C)\,(C-A)\,(A-B)}{2k},$$ k being the area of the triangle. The envelope of the lines in which $\lambda\,.\,ef+\mu\,.\,fd+\nu\,.\,fe=0$ is a parabola: λ, μ, ν being as the coordinates of its axis. This equation can be written in such shapes, that an orthogonal substitution gives the envelope of the tangent at the vertex, and their form shows that they are also the envelope of the asymptotes of the rectangular hyperbolas circumscribing the triangle. The intercept of such a tangent between two sides of the triangle is equal to the projection of the remaining side upon it. 45

1709. 1. Let $f(x)$ be any function of x, and let $f(a, b)$ express (according as a is greater or less than b) a superior or inferior limit to $f(x)$ for all values of x between a and b; show that if $f(a, b)$ has the same sign as $b-a$, no real root of the equation $f(x)=0$ exists between a and b.

2. Let $f(x)=A_n x^n + A_{n-1}x^{n-1}+ \ldots\ldots + A_1 x + A_0$, n being a positive integer. Show that, when a and b are positive, $f(a, b)$ may be found by the following process. According as A_n is positive or negative, multiply it by a or b; add the product to A^{n-1}, and represent the result by k_1. According as k_1 is positive or negative, multiply it by a or b; add the product to A_{n-2} and represent the result by k_2. Proceeding in this way till $k_1, k_2, k_3 \ldots\ldots k_n$ are all found, we shall have $f(a, b)=k_n$. 57

1710. Trace the curve $y^4-2y^2zx-z^4=0$, where the coordinates are such that $x+y+z=0$ is the line *infinity*. 17

1711. To a uniform beam supported at its extremities is suspended a weight equal to that of the beam: show that, if the point of suspension is nearer to the centre than to an end of the beam, the moment of strain will be greatest at that point, but otherwise greatest at a point at the same distance from the centre, as the point of suspension is from the end of the beam. 29

1712. Let $f(x)$ be any rational integral function of (x) of the degree n, and $f'(x)$ the first derived function; then will $f'(x_1)\,.\,f'(x_2)\ldots\ldots f'(x_n)=a^{2-n}\,n^n\,\Delta$, where a is the coefficient of the highest power of x in $f(x)$, Δ is the discriminant, and $x_1, x_2 \ldots\ldots x_n$ are the roots of $f(x)=0$. 50

1715. Let P be the point of intersection of the perpendiculars of any triangle ABC, O the centre of the circumscribed circle, M the middle point of PO and O′ the centre of any one of the escribed circles of the triangles ABC, BPC, CPA, APB. Prove that the nine-point circle of the above triangle touches the common tangent of the circle (O′) and the circle described on MO′ as diameter. 60

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1717. Show how the locus of the foci of conics, subject to four conditions, may be found in the following cases:—(1.) When each conic of the system touches three lines, and satisfies any fourth condition. (2.) When each conic passes through three fixed points, and satisfies any fourth condition. (3.) When a given triangle is self-conjugate with regard to each conic of the system, which also satisfies any fourth condition. 18

1721. Find the locus of a point such that, if perpendiculars are drawn therefrom on the sides of a quadrilateral, a circle will pass through the feet of these perpendiculars; and show geometrically how to find points on the locus; also determine the quadrilateral in order that the locus may be a conic. ... 40

1724. The equations of three conics being given in the following forms: viz.,

$a_1x^2+b_1y^2+c_1z^2+d_1w^2=0,\quad a_2x^2+\&c.=0,\quad a_3x^2+\&c.=0,$

where $x+y+z+w\equiv 0$, show that a straight line $(\xi x+\eta y+\zeta z+\omega w=0)$ will be cut in involution by them if

$$\Sigma.\begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \end{vmatrix}.(\xi-\eta)(\xi-\zeta)(\xi-\omega).\text{(to four terms)}=0.$$ 52

1726. Let O_1, O_2, O_3 be the centres of the escribed circles having internal contact with the sides BC, CA, AB respectively of the triangle ABC, and P any other point in the same plane; then (1) if R_1, R_2, R_3 be the intersections of AP, BP, CP with BC, CA, AB, respectively, the triad (O_1R_1, O_2R_2, O_3R_3) will be concurrent; and (2) if S_1, S_2, S_3 be the intersections of O_1P, O_2P, O_3P with BC, CA, AB, respectively, the triad (AS_1, BS_2, CS_3) will be concurrent. 93

1727. If the middle points of the sides of any triangle be joined, and the middle points of the sides of this new triangle also joined, and so on *ad infinitum*; the limit to which these triangles tend is a point, and the sum of the squares on the lines drawn therefrom to the angles of all the inscribed triangles is one-third of the sum of the squares on the lines drawn from the same point to the angles of the original triangle. 76

1732. Prove that the characteristics (see Question 1573) of a system of conics, satisfying four conditions, remain unaltered when, in place of passing through a given point, each conic is required to divide a given finite segment harmonically. 56

1733. To find the area of a triangle the equations of whose sides in trilinear coordinates are

$l_1\alpha+m_1\beta+n_1\gamma=0,\quad l_2\alpha+m_2\beta+n_2\gamma=0,\quad l_3\alpha+m_3\beta+n_3\gamma=0.$ 53

1737. Solve the equations

$\alpha xy+\alpha' xz+a=0,\quad \beta yz+\beta' yx+b=0,\quad \gamma zx+\gamma' zy+c=0.$ ... 50

1739. Let $f(x)=A_nx^n+A_{n-1}x^{n-1}+\ldots+A_1x+A_0$. Let $f_1(x)=A_{n-1}x^{n-1}+2A_{n-2}x^{n-2}+\ldots+(n-1)A_1+nA_0$. Form $f_2(x)$ from $f_1(x)$ in the same manner, and so on to $f_n(x)$. Let a

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and b be any two quantities *of the same sign*, of which a is greater numerically than b. Show that if the series of functions $f(x)$, $f_1(x)$, $f_2(x) \ldots f_n(x)$, give p changes of signs when $x = a$, and q changes when $x = b$, the number of real roots of $f(x) = 0$ between a and b cannot exceed $p-q$. 102

1741. Let P be the point of intersection of the perpendiculars, and G the centre of gravity, of any triangle ABC; prove that the nine-point circles of all the triangles that can be formed from the points A, B, C, P, G, together with the maximum ellipse that can be inscribed in the triangle ABC, all pass through a common point, viz., the centre of the equilateral hyperbola, passing through A, B, C, P, G. 89

1743. If P, Q be two points in the path of a projectile; PT, QT the tangents at P, Q; and α, β, γ the angles made with the horizon by PT, TQ, QP; prove geometrically that

$$\tan\alpha + \tan\beta = 2\tan\gamma.$$

2. Hence show that if a particle be projected from one end of the horizontal base of an isosceles triangle so as exactly to shoot down the opposite side; and if α be the angle of projection, γ the angle at the base of the triangle; then

$$\tan\alpha = 3\tan\gamma.$$

3. Prove that the velocities at P, Q are proportional to PT, QT. 100

1744. It is required to find (x_1, y_1, z_1), functions of (x, y, z), such that we may have identically

$$\frac{x_1^3+y_1^3+z_1^3}{x_1y_1z_1} = \frac{x^3+y^3+z^3}{xyz}.$$ 32

1749. AB is a diameter of a circle, O its centre, and P a point in OA, such that $OP = \frac{1}{3}OA$. Through P any straight line QPR is drawn, meeting the circumference in Q, R; and the straight line passing through Q, O is produced to meet that through B, R in S.

If $\angle OSB = \psi$, and $\angle OBS = \phi$, prove that $\dfrac{1-\sin\psi}{1+\sin\psi} = \left(\dfrac{1-\sin\phi}{1+\sin\phi}\right)^3$. 20

1750. Given four straight lines whose equations are connected by the syzygy $x+y+z+w=0$; show that the straight lines

$$lx+my+nz+sw=0, \quad \lambda x+\mu y+\nu z+\sigma w=0,$$

will be conjugates in respect of any conic touching (x, y, z, w), if $(l\mu+\lambda m)+n\sigma+\nu s) = l\nu+\lambda n)+(su+\sigma m) = (l\sigma+\lambda s)+(m\nu+\mu n)$. Show also that if a quadrangle be formed from the quadrilateral by taking the pole of each line in respect of the triangle formed by the other three, then the relation between the two figures will be reciprocal; and if two straight lines be conjugates in respect of any conic inscribed in the quadrilateral, their poles in respect of the common connector-triangle will be conjugates in respect of any conic circumscribing the quadrangle. 63

1751. Let ABCD be any quadrilateral. Construct, as shown in the figure, the points F, G, H, I: in BC find a point Q such that $\dfrac{BG}{BC}\cdot\dfrac{CQ}{GQ} = \dfrac{1}{\sqrt{2}}$; and complete the construction as

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shown in the figure. Show that an ellipse may be drawn passing through the eight points F, G, H, I, α, β, γ, δ, and having at these points respectively the tangents shown in the figure. 38

1752. Prove that any proper fraction may be expanded under the form $\frac{e_1}{2}+\frac{e_2}{2^2}+\frac{e_3}{2^3}+\ldots\ldots\frac{e_n}{2^n}+\ldots\ldots$ where the e's are all either zero or unity, and moreover become periodic after a certain point in the series; and show thereby that a calculating machine capable of multiplying and extracting square roots, will serve for the extraction of any root whatever. 64

1753. The tangential equation of the surface of the centres of curvature of an ellipsoid being

$$(\xi^2+v^2+\zeta^2)=\left(\frac{\xi^2}{a^2}+\frac{v^2}{b^2}+\frac{\zeta^2}{c^2}\right)(a^2\xi^2+b^2v+c^2\zeta^2-1);$$

show that any two parallel tangent planes being drawn to the surface of centres and to the ellipsoid, the difference of the squares of the coincident perpendiculars let fall upon them from the common centre is always equal to the square of the coinciding semidiameter of the ellipsoid 115

1754. If a triangle and its nine-point circle are drawn; in the parabola which touches two sides and the perpendiculars to them through the opposite angles, the focus, the directrix, and the tangents to it from the centre of the circle are given by the figure........ 114

1756. If normals drawn to a central quadric at the points (1), (2), (3), (4), (5), (6) pass through a point, and r_{12} denote the length of the semi-diameter parallel to the chord (1, 2); prove that $r_{12}^2+r_{34}^2+r_{56}^2=a^2+b^2+c^2$, where a, b, c are the semi-axes of the quadric 99

1760. Let each angular point of any triangle ABC be joined with each of two given points: and let α, β, γ; α', β', γ' denote the six points of intersection of the joining lines with the opposite sides of the triangle. It is required to prove that the conic which touches the sides of the two triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ touches also the two axes of homology of the triangles ABC, $\alpha\beta\gamma$, and ABC, $\alpha'\beta'\gamma'$. Show also that the triangle ABC is self-conjugate with respect to this conic. What is the reciprocal of this theorem? 65

1761. Let $f(x)=0$ be any equation of the nth degree. Let $f^r(x)$ be the rth derived function of $f(x)$. Let it be known that $f^r(x)=0$ has only one root between the two quantities a and b. Show that by giving a more general expression to the trial fraction in Horner's method of approximation we can simultaneously approximate as nearly as we please to the root of $f^r(x)=0$, separate any two roots of $f^{r-1}(x)=0$ that may exist between a and b, and obtain information as to the number of roots of $f^{r-2}(x)=0$, $f^{r-3}(x=0, \ldots f^1(x)=0$, $f(x)=0$ passed over in the operation. Show also

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that the same process will enable us to approximate simultaneously to any maximum or minimum value of $f(x)$, and to the value of x, which makes $f(x)$ a maximum or minimum. 102

1765. Two weights are attached by short strings of given length to the surface of a cylinder; determine the position of equilibrium of the cylinder when resting on a horizontal plane, the weights resting on the surface of the cylinder 86

1766. To find the condition necessary in order that one of two homofocal ellipses should be inscribed in a triangle, and the other circumscribed about it.. 49

1771. Given a circle and a line, it is required to find a parabola, having the line for its directrix, and the circle for a circle of curvature.. 70

1773. A traveller starts from a point on a straight river, and travels in a random direction a certain distance in a straight line. Having quite lost his way, he starts again at random the next morning, and travels the same distance; what is the chance of his reaching the river again in the second day's journey? 71

1774. If the ends of a fixed diameter of a circle be joined to the ends of a chord variable in position, but of given magnitude, the locus of the centre of the circle circumscribing the triangle formed by the chord and the segments of the joining lines outside the circle, is a circle concentric with a given one. 72

1775. If a straight line meet the faces of the tetrahedron ABCD in the points a, b, c, d, respectively; the spheres whose diameters are Aa, Bb, Cc, Dd have a common radical axis. 65

1780. Find the locus of a point whose polar, in reference to a given conic, cuts off a given area from the axes. 68

1781. From a point P in the produced major axis of an ellipse to draw a line PQR, cutting the curve in Q and R, so that if Q and R be joined with the centre O of the ellipse, the triangle OQR may be a maximum..................................... 69

1784. If from any given point in an ellipse there be drawn an ordinate to either axis, and if from the point where this meets the curve again a diameter be drawn, and to this diameter another ordinate be applied from the given point; then the point in which this ordinate meets the ellipse shall be in the circumference of the circle of curvature at the given point. 97

1786. If $4a$ be the parameter of a parabola, S_y the length (measured from the vertex) of an arc the ordinate of whose extremity is y, and R_y the radius of curvature at the extremity, prove that $S_{4a} - 3S_a = R_a$. 69

1789. Let abc be the triangle formed by the three diagonals AA′, BB′, CC′ of a quadrilateral: and let Aa, B′b, C′c meet A′BC in $\alpha_1, \alpha_2, \alpha_3$; A′$a$, B′$b$, C′$c$ meet AB′C in $\beta_1, \beta_2, \beta_3$; A′$a$, B′$b$, C$c$,

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meet ABC′ in $\gamma_1, \gamma_2, \gamma_3$; Aa, Bb, Cc, meet A′B′C in $\delta_1, \delta_2, \delta_3$; then $\alpha_2, \beta_2, \gamma_2, \delta_2$, are the points of contact of the "*Critical* Inscribed Conic" of the quadrilateral CA′C′A (this may be taken as the *Definition* of the "*Critical* Inscribed Conic"), and Professor Cayley has given in Quest. 1751 (*Reprint*, Vol. IV., p. 38), a construction for finding four other points on the conic and the tangents thereat. It is required to show that, by the ruler alone, *eight* other points on this conic, and the tangents thereat, may be found by the following

Construction.—Let $\alpha_2\beta_2$ meet C′A′ in p; join $p\gamma_3$ cutting $C\delta_2$ in P; then P is a point on the curve, and Pp the tangent at P. Similarly, *seven* other points may be found, and the tangents constructed thereat 87

1790. (1.) If a set of six points be respectively represented by the six permutations of $a : b : c$, show that they lie in a conic, and write down its equation.

(2.) *Hence* prove that if AB, BC, CA be three real lines containing the nine points of inflexion of a cubic curve having an oval, the pairs of tangents drawn to the oval from A, B, C will meet it in six points lying in a conic 81

1795. (1) Let P be the point in a homogeneous triangular lamina ABC at which the sides subtend equal angles. Show that if the lamina be placed in a smooth prolate spheroid whose long axis is vertical, it will rest in equilibrium when the point P coincides with the lower focus of the spheroid.

(2) If the lamina be not homogeneous, and its centre of gravity be given, construct for the corresponding position of the point P. 115

1807. From the ends of a diameter of a semicircle, two persons walk in random directions within the semicircle; find the chance of their paths intersecting inside the circle. 92

1811. (α). Prove that the chance of three points taken at random within a circle or sphere forming the vertices of a triangle of any prescribed form is the same as if two of the points are restrained to move on the circumference or surface, with a likelihood varying as the square of their distance apart.

(β). By the aid of this principle show that the chance of a triangle whose angular points are taken at random within a circle being acute is

$$\int_0^{\frac{1}{3}\pi} (3 \sin^3\theta \cos\theta + \theta \sin^2\theta - \tfrac{1}{3}\pi \sin^4\theta)\, d\theta + \int_0^{\frac{1}{3}\pi} \sin^2\theta\, d\theta, \text{ or } \frac{4}{\pi^2} - \frac{1}{8};$$

and that the chance of a triangle whose angular points are taken at random within a sphere being obtuse is

$$\int_0^{\frac{1}{3}\pi} \sin^3 \cos\theta\,(6 - 6\sin\theta - 3\cos\theta + 4\sin^3\theta + \cos^3\theta)\, d\theta \text{ or } \frac{37}{70}.$$

(γ). Find also the chance that no angle of the triangle will exceed any magnitude α between $\frac{1}{3}\pi$ and π, distinguishing between the case where α is greater and that where it is less than $\frac{1}{2}\pi$. 111

No Page

1812. Find the envelope of a series of circles which touch a given straight line and have their centres in the circumference of a given circle. 107

1814. A triangle is inscribed in an ellipse and envelopes a confocal ellipse: prove that the points of contact lie on the escribed circles........ 108

1816. Express the roots of the equation

$$(ae-4bd+3c^2)(ax^4+4bx^3+6cx^2+4dx+e)^2-3\{(ac-b^2)x^4+2(ad-bc)x^3$$
$$+(ae+2bd-3c^2)x^2+2(be-cd)x+(ce-d^2)\}^2=0,$$

in terms of the roots $\alpha, \beta, \gamma, \delta$ of $x^4+4bx^3+6cx^2+4dx+e=0$. 108

1823. The conicoids which pass through six fixed points in space intersect any plane in a series of conics having a common self-conjugate quadrilateral. *Any four* conics have a common self-conjugate quadrilatral. (DEF.—A quadrilateral is *self-conjugate* in respect of a conic which divides its diagonals harmonically) 109

1828. Exhibit the expression

$$(\beta-\gamma)^2(\beta+\gamma-2\alpha)+(\gamma-\alpha)^2(\gamma+\alpha-2\beta)+(\alpha-\beta)^2(\alpha+\beta-2\gamma)$$

(1) as the product of three factors, (2) as the sum of two cubes 110

1829. Two points being taken at random within (1) a circle or (2) a sphere, what is the probability that their distance apart is less than a given line? 77

1847. Prove that the method given by Tschirnhäusen for removing the second and fourth terms from the binary quartic $(a, b, c, d, e)(x, y)^4=0$ depends ultimately on the solution of the equation $\lambda^3-I\lambda+2J=0$,

where $I=ae-4bd+3c^2$, and $J=ace+2bcd-ad^2-eb^2-c^3$... 111

Unsolved Questions.

1480. (Proposed by Professor SYLVESTER.)—Prove that if through the middle point of either diagonal of any of the three quadrilateral faces of a tetrahedral frustum, and the middle points of the two edges which meet but are not in the same face with that diagonal, a plane be drawn, the six planes thus obtained will touch the same cone.

1481. (Proposed by Professor HIRST.)—Find the envelope of a conic which circumscribes a given triangle, and is cut harmonically by two fixed straight lines.

N.B.—A conic is said to be cut harmonically in two pairs of points, when the tangents at those points cut every other tangent harmonically, or, what amounts to the same thing, when the connectors of those points with any other point on the conic form a harmonic pencil.

No.

1495. (Proposed by HUGH GODFRAY, M.A.)—Show that $\frac{1}{8}n(n-1)(n-2)$ points can always be so arranged in a plane that they shall be situated by eights in $\frac{1}{24}n(n-1)(n-2)(n-3)$ circles.

1496. (Proposed by MATTHEW COLLINS, B.A.)—Prove that a triangular pyramid whose vertices are ABCD, and a parallelepiped formed from it as follows, have the same centre of gravity; viz., through any point in each of the opposite edges AB and CD draw straight lines parallel to the other edge, we thus get two parallel plane faces of the parallelepiped; two other parallel faces of it are similarly obtained from the opposite edges AC and BD; and the third pair of faces are obtained from the remaining two opposite edges AD and BC.

1503. (Proposed by Professor SYLVESTER.)—A table has n holes bored in its rim, into which ν pegs are to be inserted at random, ν being not greater than $\frac{1}{2}n$. Show that the probability of there being no two pegs without one or more unoccupied holes between them will be equal to $\dfrac{\pi(n-\nu)\,.\,\pi(n-\nu-1)}{\pi(n-1)\,.\,\pi(n-2\nu)}$, and, if ν is given, approaches to certainty as n becomes indefinitely great.

[$\pi(n)$ here denotes the product $1\,.\,2\,.\,3\,\ldots\ldots n$.]

1504. (Proposed by Professor SYLVESTER.)—Trace the two sextic curves

$$y+27x^2-28x^3=x^2\sqrt{(72x-80x^2)} \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\frac{y^{\frac{1}{3}}}{6\sqrt{3}}=(1-x)+(1-x)^{\frac{3}{2}} \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and prove that they intersect in two real points and touch in a third point which are to be found.

1511. (Proposed by Professor SYLVESTER.)—At a dancing and card party lately held at the Palazzo Finto, l ladies and g gentlemen sat down miscellaneously at a table to play a round game; when, on its being announced that more dancers were wanted, every cavalier who had a lady sitting next to him on his right hand, gave her his arm to lead her into the ball-room. Prove that the number of couples which there is the *greatest* probability of having been thus formed is the integer part of the quotient of lg by $(l+g+1)$, if the division leaves a *remainder;* but if there is *no remainder*, then indifferently that number or the number next below it.

1515. (Proposed by the late H. J. PURKISS, B.A.)—Find the conditions that the general equation of the second degree in tetrahedral coordinates may represent a surface of revolution.

1554. (Proposed by Professor CAYLEY.)—Show that in the ellipse and its circles of maximum and minimum curvature respectively, the semi-ordinates through the focus of the ellipse are,

for the circle of maximum curvature, $y=a(1-e)(1+2e)^{\frac{1}{2}}$;

for the ellipse, $y=a(1-e^2)$;

for the circle of minimum curvature, $y=\dfrac{a\left\{1-e^2+e^4)^{\frac{1}{2}}+e^2\right\}}{(1-e^2)^{\frac{1}{2}}}$;

and that these values are in the order of increasing magnitude.

No.

1587. (Proposed by the Rev. T. P. KIRKMAN, F.R.S.)—Apollo and the Muses accepted the challenge of Jove, to vary the arrangement of themselves on their fixed and burnished couches at his evening banquets, till every three of them should have occupied, once and once only, every three of the couches, in every and any order. In how many days, and how many ways, did they accomplish the feat, keeping one arrangement of themselves through all the solutions? Required two or more of those solutions, clearly indicated, so as to save space, by cyclical operations.

1604. (Proposed by Professor SYLVESTER.)—Find the area, centre of gravity, and principal axes of the segment cut off from any conic by any straight line; the whole expressed in trilinear coordinates.

1611. (Proposed by Dr. BOOTH, F.R.S.) — The inverse curve of the ellipse $a^2y^2+b^2x^2 = a^2b^2$, the centre being the pole and $k^2=ab$, is $a^2y^2+b^2x^2 = (x^2+y^2)^2$. Rectify this curve, and show that its length is equal to that of a spherical ellipse, the symmetrical intersection of a sphere whose radius is $\sqrt{(a^2+b^2)}$ with a right cylinder standing on the original ellipse.

1625. (Proposed by Professor CREMONA.)—If from a point on a cubic four straight lines be drawn to touch the curve elsewhere, three of the latter will be the polar lines of the point of contact on the fourth; relative, respectively, to the three cubics whose common Hessian is the given one.

1634. (Proposed by W. A. WHITWORTH, M.A.)—Required the geometrical *interpretation* of the following property:—

If $(\alpha', \beta', \gamma')$ be any point on a conic whose equation in trilinear coordinates is $f(\alpha, \beta, \gamma) = 0$, then the condition that $f(\alpha+i\alpha', \beta+i\beta', \gamma+i\gamma')$ should be a real quantity, being expressed algebraically, is the equation to the tangent to the conic at the point $(\alpha', \beta', \gamma')$.

[i denotes, as usual, the square root of negative unity, $\sqrt{(-1)}$.]

1637. (Proposed by Professor HIRST.)—If a, b, c be fixed points on cubic, what will be the envelope of the line joining a variable point v thereon to the opposite of the four points a, b, c, v?

1640. (Proposed by A. RENSHAW.)—Given four points A, B, C, D in a plane. Find the locus of a fifth point P in the plane, such that $PA \pm PB = PC \pm PD$. What will the locus be when the system is not restricted to a plane?

1641. (Proposed by E. McCORMICK.)—An ellipse is placed with its major axis vertical; find geometrically the straight line of quickest descent from the upper focus to the curve.

1674. (Proposed by J. W. T. BLAKEMORE, B.A.)—A cylinder filled with fluid is closed at both ends, and then suspended from a point in the rim of one end; prove that the direction of the resultant pressure on the curved surface, and the axis of the cylinder, make equal angles with the vertical and horizontal respectively.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

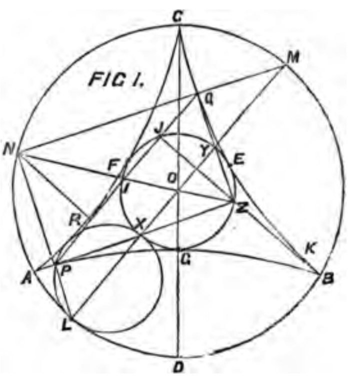
ON THE ENVELOPE IN QUESTION 1679.

By the REV. R. TOWNSEND, M.A.

Nearly all the interesting results enunciated by Steiner in his paper on this Envelope (see the *Educational Times* for June) may be readily deduced from the two following properties of the hypocycloid of three cusps:

(I.) *The chord intercepted by the inscribed circle on any tangent to the curve is trisected externally at its point of contact with the curve.*

(II.) *The two arcs into which the inscribed circle is divided by any tangent to the curve are trisected internally at their points of contact with the corresponding branches of the curve.*



These may be proved as follows:— Let A, B, C (Fig. 1) be the positions of the three cusps on the circumscribed circle; E, F, G those of the points of contact of the three opposite branches with the inscribed circle; O the common centre of both circles; L any point on the former circle; P the corresponding point on the curve; and XZ the chord intercepted by the latter circle on the tangent at P, which, being perpendicular to PL, passes, as is well known, through the point of contact X of the generating and inscribed circles. Then, since, from the equality of the two latter circles, chord XZ = chord XP, there-

fore length $PZ = 2$ length PX, which proves (I.); and since, from the equality of the two arcs XP and LD, and consequently of the two arcs XZ and LD, arc XZ : arc XG = arc LD : arc XG = rad. OD : rad. $OG = 3 : 1$, therefore (II.) follows; the trisection of the arc XFZ at F being, of course, necessarily involved in that of the arc XGZ at G.

N.B.—By virtue of (I.), the point X was termed by Steiner the *centre* of the tangent PZ; and the point Z, for distinction, the *vertex* of the same. These terms are convenient, and may be employed when required.

It follows of course, conversely, from the above, that, *If a variable arc* XGZ *of a fixed circle have a fixed point of internal trisection* G, *its chord* XZ *envelopes a tricusped hypocycloid, which touches the circle at* G, *and at the two other points* E *and* F *which with* G *trisect the circumference; and the chord itself, in every position, is trisected externally at its point of contact with the envelope.* These are evident from the direct properties.

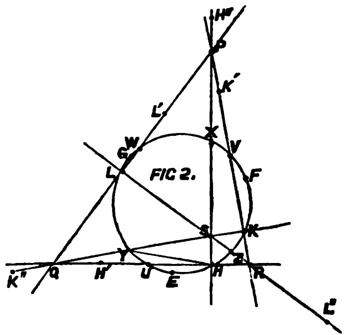
Cor. 1.—If a chord ZX of the inscribed circle touch any branch of the curve at P, the perpendicular chord ZY at its vertex Z touches another of its branches at Q, and the line PQ, connecting the two points of contact P and Q, touches the third branch at R. Also, the length of the connecting line PQ is constant, whatever be the position of the first tangent ZX, and equal to twice the diameter of the inscribed circle XY; and the three normals to the curve at the three points of contact P, Q, R intersect at a common point N on the circumscribing circle, diametrically opposite to the position of Z.

For, if the arc XGZ intercepted by XZ be trisected, as it is by hypothesis, at G, then manifestly the arc YEZ intercepted by YZ is trisected at E; and as the line PQ joining the points of contact P and Q of the two tangents ZX and ZY is then parallel to the diameter XY and at double its distance from the point Z, the arc IFJ which it intercepts is trisected at F; which proves the first part. And, as the two normals PL and QM, at the two points P and Q of the curve, form with the two tangents PZ and QZ at those points a rectangle PZQN, whose diagonals bisect each other, therefore their point of concurrence N is on the circumscribing circle and diametrically opposite to Z; and, the two triangles NIR and ZIJ being equal in all respects, as the angle at J is a right angle, so also is the angle at R, or the line NR is the normal at the point of contact R of the third tangent PQ, whose length intercepted between the two points P and Q is constant and equal to twice that of the diameter XY; which proves the remainder.

N.B.—Since PNQ is a right angle, it follows that the two points L and M on the circumscribing circle, corresponding to the points of contact P and Q of the two rectangular tangents ZX and ZY, are diametrically opposite points of that circle.

As the tricusped hypocycloid manifestly admits in no direction of two real tangents parallel to each other, it follows consequently from the above, that *In a tricusped hypocycloid, when two tangents intersect at right angles,* (*a*) *they belong to different branches,* (*b*) *they correspond to diametrically opposite points on the circumscribed circle,* (*c*) *they intersect on the inscribed circle,* (*d*) *their chord of contact is of constant length,* (*e*) *and is a tangent to the third branch,* (*f*) *their corresponding normals intersect on the circumscribed circle,* (*g*) *and are concurrent with the normal corresponding to their chord of contact.*

It follows from this last, of course, conversely, that, in a tricusped hypocycloid, (*a*) the segment intercepted by any two of the three branches on any tangent to the third is of constant length, (*b*) the tangents to those branches at its extremities intersect at right angles on the inscribed circle, (*c*) the normals corresponding to the three tangents concur to a point on the circumscribed circle.



Cor. 2.—If HU and HX (Fig. 2) be any pair of rectangular tangents intersecting at H on the inscribed circle, and QS the segment they intercept on any third tangent KY, the segment QS is always bisected by the circle at the centre Y of the tangent; and conversely, if the circle bisect the segment of a line intercepted by any pair of rectangular tangents to the curve, the line is a tangent, and the point of bisection its centre.

For since, by (II.), when KY is a tangent, arc EK = 2 arc EY, and arc EH = 2 arc EU, therefore arc HK = 2 arc UY, and consequently angle HYS = 2 angle YHQ; and therefore the first part follows, since QHS is a right angle. Conversely, when QS is bisected at Y, then, QHS being a right angle, HYK = 2UHY, and therefore arc EK = 2 arc EY; and therefore, by (II.), the remaining part follows.

N.B.—If, while the pair of rectangular tangents HU and HX are supposed to remain fixed, the third tangent KY be conceived to vary, then, since in every position the triangle HYQ is isosceles, and the two bisectors of its vertical angle Y consequently parallel in every position to the fixed pair of tangents HU and HX, it follows from the above that *if while the vertex* Y *of a variable angle* HYQ *moves on a fixed circle, one of its sides* HY *turns round a fixed point* H *on the circle, and its bisectors are constantly parallel to two fixed rectangular chords* HU *and* HX *passing through the point, the other side* YK *will envelop a tricusped hypocycloid which touches the circle at the three points of trisection* E, F, G *of the four arcs intercepted by the two chords.*

Cor. 3.—If P and Q, R and S, (Fig. 2.) be the two pairs of opposite intersections of any two different pairs of rectangular tangents HU and HX, KV and KY, their two lines of connection PQ and RS determine a third pair of rectangular tangents LW and LZ. For, the two pairs of lines HU and HX, KV and KY, being, by hypothesis, rectangular, the four points P, Q, R, S determine a tetrastigm whose three pairs of opposite connectors, QR and PS, RP and QS, PQ and RS, are rectangular; and since the inscribed circle, which, by Cor. 1, passes through the two points of intersection H and K, passes also, by Cor. 2, through the four points of bisection U and X, V and Y of the two pairs of connectors QR and PS, RP and QS, it is consequently the nine-point circle of the tetrastigm, and therefore passes through the point of intersection L, and through the two points of bisection W and Z of the remaining pair of connectors PQ and RS, which connectors are consequently, by Cor. 2, tangents to the figure.

N.B.—Of the four points P, Q, R, S, through each of which three concurrent tangents pass, and each of which is the polar centre of the triangle determined by the remaining three, each, taken by itself, irrespectively of the others, being of course entirely arbitrary, it follows consequently from the above, that, *when three tangents to a tricusped hypocycloid are concurrent, the three rectangular tangents determine a triangle, of which the point of concurrence is the polar centre, and of which the inscribed circle of the figure is the nine-point circle.*

COR. 4.—If H and H″, K′ and K″, L′ and L″, (Fig. 2.) be the six points of contact of the three pairs of tangents HU and HX, KV and KY, LW and LZ, then, since, by (I.), the six segments HH′ and HH″, KK′ and KK″, LL′ and LL″, are concentric with the six QR and PS, RP and QS, PQ and RS, at the six points U and X, V and Y, W and Z respectively, and since, in the four triangles PQR, QSR, RSP, PSQ, the three perpendiculars to the sides at the three points H, K, L are concurrent, therefore in the same (*Mod. Geom.* Art 133, Ex. 6), the four triads of perpendiculars to the sides at the four triads of points H′, K′, L′; L″, H′, K″; H″, K′, L″; K″, L′, H″, are concurrent, and the four points of concurrence connect evidently with the four corresponding points S, P, Q, R respectively, by four lines passing through the centre of the common nine-point circle, which point divides them all in the common ratio of 3 : 1. Hence, *when three tangents to a tricusped hypocycloid are concurrent, the three normals at the points of contact of the three rectangular tangents are also concurrent, and the two points of concurrence connect through the centre of the inscribed circle of the figure from which point they are distant in opposite directions in the constant ratio of* 3 : 1.

N.B.—Fig. 1. supplies an example of a particular case of the above, the three tangents ZP, ZQ, ZK being concurrent at Z, and the normals QM, PL, RN, corresponding to the three perpendicular tangents ZQ, ZP, PQ, being also concurrent at N; also, the line NZ passing through O, and being there divided so that ON : OZ = 3 : 1.

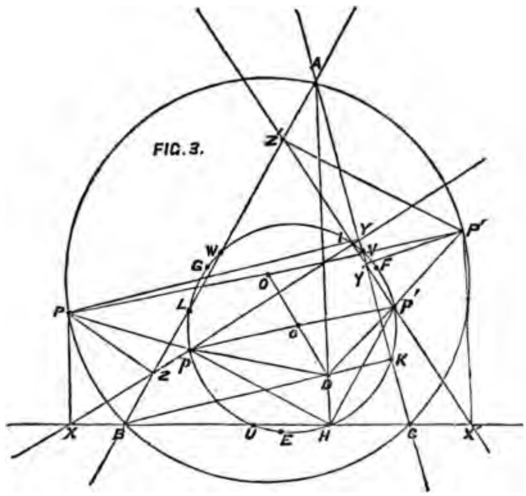
COR. 5.—In the same four triangles PQR, QSR, RSP, PSQ, (Fig. 2.), the four triads of connectors PH, QK, RL; QL, RK, SH; RH, PL, SK; PK, QH, SL being concurrent, therefore in the same (*Mod. Geom.* Art 137, Ex. 11.) the four triads of connectors PH′, QK′, RL′; QL″, RK″, SH′; RH″, PL″, SK′; PK″, QH″, SL′ are concurrent, and therefore, *when three tangents to a tricusped hypocycloid are concurrent, the three rectangular tangents determine a triangle whose vertices connect concurrently with the points of contact of the opposite sides.*

N.B.—From the converse of this latter property combined with that of the preceding corollary it follows immediately that, *if a conic have triple contact with a tricusped hypocycloid, the three normals at the three points of contact are concurrent; and the line connecting their point of concurrence with the polar centre of the triangle determined by the three tangents passes through the centre of the figure, and is there divided in the constant ratio of* 3 : 1.

COR. 6.—As every equilateral hyperbola which passes through any three of the four points P, Q, R, S, (Fig. 2.) passes, as is well known, through the fourth, and has its centre on the nine-point circle of the tetrastigm they determine; and as the pair of rectangular tangents to the figure, which intersect at that or any other point on that circle, intercept (by Cor. 2.) on the six lines of that tetrastigm six segments concentric with the six connectors QR and PS, RP and QS, PQ and RS; it follows consequently that those tangents are the asymptotes of the hyperbola having its centre at their point of intersection, and therefore that, *when four points are such that each is the polar centre of the triangle determined by the remaining three, the asymptotes of every equilateral hyperbola which passes through any three of them (and therefore through the fourth) are tangents to the tricusped hypocycloid which touches the three pairs of opposite connectors of the tetrastigm they determine.*

COR. 7.—If A, B, C (Fig. 3) be the three vertices of any triangle; D its polar centre; P any point on its circumscribing circle; P′ the diametrically

opposite point; X, Y, Z the feet of the three perpendiculars from P, and X′, Y′, Z′ those of the three from P′ on its three sides BC, CA, AB, respectively; the two lines of collinearity XYZ and X′Y′Z′ intersect at right angles on its nine-point circle, and are tangents to the tricusped hypocycloid which touches its three sides and perpendiculars, and consequently also touches that circle at the three points of trisection E, F, G, common to the six pairs of arcs into which it is divided by the six lines.



For if pop' be the diameter of the nine-point circle parallel to the diameter POP′ of the circumscribing circle; and H the foot of any one of the three perpendiculars AH, BK, CL of the triangle; then, the polar centre D being the external centre of similitude of the two circles, the two lines DP and DP′ pass through and are bisected at p and p', which therefore (see Solution to Quest. 1649) are points on the two lines XYZ and X′Y′Z′; hence the two triangles XpH and X′p'H are isosceles, and therefore (Cor. 2, Note) the two lines XYZ and X′Y′Z are tangents to the hypocycloid in question; and as their centres p and p' are diametrically opposite points of its inscribed circle, they therefore (Cor. 1) intersect at right angles on that circle.

N.B.—From the converse of this latter property, combined with that of the preceding Corollary, it follows, indirectly, that *the two triads of perpendiculars to the three sides of a triangle at the two triads of points at which they are met by the asymptotes of any equilateral hyperbola passing through its three vertices concur to two diametrically opposite points on its circumscribing circle*—a property which may be easily proved directly.

1710. (Proposed by Professor CAYLEY.)—Trace the curve $y^4 - 2y^2zx - z^4 = 0$, where the coordinates are such that $x + y + z = 0$ is the line *infinity*.

Solution by the PROPOSER.

We have $x = \dfrac{y^4 - z^4}{2y^2 z}$; or writing $y = \theta z$, then $x = \dfrac{\theta^4 - 1}{2\theta^2} z$, that is

$$x : y : z = \theta^4 - 1 : 2\theta^3 : 2\theta^2.$$

Hence, we see that y, z are indefinitely small in comparison of x,

if $\theta = \infty$, and then $x : y : z = \theta^4 : 2\theta^3 : 2\theta^2$, that is $y^2 = 2zx$;

or, if $\theta = 0$, and then $x : y : z = -1 : 2\theta^3 : 2\theta^2$, that is $z^3 = -2y^2 x$;

so that in the neighbourhood of the point ($y = 0$, $z = 0$) there are two branches coinciding with the parabola $y^2 = 2zx$ and with the semicubical parabola $z^3 = -2y^2 xz$, respectively.

For the points at infinity we have $x + y + z = 0$, that is $\theta^4 + 2\theta^3 + 2\theta^2 - 1 = (\theta + 1)(\theta^3 + \theta^2 + \theta - 1) = 0$; and observing that the equation $\theta^3 + \theta^2 + \theta - 1 = 0$ has one real root, say $\theta = k$, if k be the real root of the equation $k^3 + k^2 + k - 1 = 0$ ($k = \cdot 505$ nearly),—there are two real points at infinity, viz., corresponding to $\theta = -1$, we have the point $(0, -1, 1)$, and corresponding to $\theta = k$ the point $(-1-k, k, 1)$.

The equation of the tangent at a point (α, β, γ) is

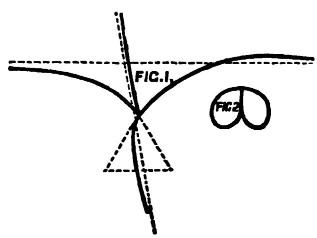
$$x(-\beta^2\gamma) + y(2\beta^3 - 2\alpha\beta\gamma) + z(-\alpha\beta^2 - 2\gamma^3) = 0,$$

and hence writing $(\alpha, \beta, \gamma) = (0, -1, 1)$ we have the asymptote $x + 2y + 2z = 0$: to find where this meets the curve, we have $\theta^4 + 4\theta^3 + 4\theta^2 - 1 = 0$, that is $(\theta + 1)^2(\theta^2 + 2\theta - 1) = 0$, or at the points of intersection $\theta^2 + 2\theta - 1 = 0$, that is $\theta = -1 \pm \sqrt{2}$, or there are two real points of intersection.

Again writing $(\alpha, \beta, \gamma) = (-1-k, k, 1)$ we find the asymptote $k^2 x - 2y + (k+1)z = 0$: to find where this meets the curve, we have $k^2(\theta^4 - 1) - 4k\theta^3 + (2k+2)\theta^2 = 0$, that is $k^2\theta^4 - 4k\theta^3 + (2k+2)\theta^2 - k^2 = (\theta - k)^2 \left\{ k^2\theta^2 - 2(k^2 + k + 1)\theta - 1 \right\} = 0$; or for the points of intersection $k^2\theta^2 - 2(k^2 + k + 1)\theta - 1 = 0$, an equation in θ with two real roots, that is the points of intersection are real.

It is now easy to lay down the curve; viz., if, to fix the ideas, the fundamental triangle is taken to be equilateral, and the coordinates x, y, z are considered to be positive for points *within* the triangle, then the curve is as shown in the annexed figure (Fig. 1.)

It may be remarked that the curve is met by every real line in two real points at least, and consequently that it is not the projection of any finite curve whatever. By a modification of the constants of the equation, we might obtain curves which are finite, such as the curve in Fig. 2; or curves with two or four infinite branches, which are the projections of such a finite curve.



1717. (Proposed by W. S. BURNSIDE, B.A.)—Show how the locus of the foci of conics, subject to four conditions, may be found in the following

cases :—(1.) When each conic of the system touches three lines, and satisfies any fourth condition. (2.) When each conic passes through three fixed points, and satisfies any fourth condition. (3.) When a given triangle is self-conjugate with regard to each conic of the system, which also satisfies any fourth condition.

Solution by T. A. Hirst, F.R.S.

If ν denote the number of conics, subject to four conditions, which touch an arbitrary right line, then according to Chasles' theory, the locus of their foci is a curve of the order 3ν, which has multiple points of the order ν at each of the circular points at infinity. This important theorem is easily demonstrated on replacing, as usual, the two circular points by any two points whatever P and Q, which have no special relation to the system of conics under consideration. We have then merely to examine the locus of the intersections of tangents from P and Q to the several conics of the system, or rather to find the number of such intersections which lie on an arbitrary line T, passing through P. Now, by hypothesis, there are ν conics which touch T, and to every such conic two tangents can be drawn from Q, so that, exclusive of the point P itself, every transversal T contains 2ν of the intersections in question. But P itself obviously represents ν such intersections, since there are ν conics of the system which touch the line PQ.

The theorem, therefore, is established, and may be at once applied to the three special cases alluded to in the question. I remark that in (1), the *characteristic* ν is obviously the number of conics inscribed in the given triangle ABC, which, besides touching an arbitrary line, satisfy the given fourth condition. The locus of the foci cuts each of the given lines, say BC, in 3ν points. But the conic inscribed to ABC and having one of its foci on BC obviously degenerates to the rectilinear connector of its two foci, of which latter one must necessarily coincide with an angle of the triangle ABC. Hence, and from the fact that A, B, and C must be similarly related to the locus under consideration, we infer that the latter, besides being of the order 3ν and circular in the order ν, has also multiple points of the latter order at the intersections of the three given lines.

In (2) and (3), the fourth conditions being alike, the characteristics ν will be equal to one another. For, as far as the characteristics are concerned, the condition that each conic of a system shall cut a given segment harmonically is equivalent to the condition that each shall pass through a given point. A demonstration of this statement of Chasles is proposed in Question 1732.

II. *Solution by the* Proposer.

1. Take as triangle of reference that formed by the three tangents; then the equation of each conic is of the form $(lx)^{\frac{1}{2}}+(my)^{\frac{1}{2}}+(nz)^{\frac{1}{2}}=0$, and the tangential equation is $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$; hence, putting Ω for $\alpha^2+\beta^2+\gamma^2-2\beta\gamma\cos A-2\gamma\alpha\cos B-2\alpha\beta\cos C$, and Σ for $2l\beta\gamma+2m\gamma\alpha+2n\alpha\beta$, if λ be so determined that $\Sigma+\lambda\Omega\equiv(\alpha x+\beta y+\gamma z)(\alpha x_1+\beta y_1+\gamma z_1)$, the points (x, y, z), (x_1, y_1, z_1) are foci (Salmon's *Conics*, chap. xviii.).

Comparing both sides of this identity, we have

$$\begin{array}{ll|ll} xx_1=\lambda & \ldots\ldots\ldots\ldots(1), & yz_1+y_1z=2(l-\lambda\cos A) & \ldots\ldots(4), \\ yy_1=\lambda & \ldots\ldots\ldots\ldots(2), & zx_1+z_1x=2(m-\lambda\cos B) & \ldots\ldots(5), \\ zz_1=\lambda & \ldots\ldots\ldots\ldots(3), & xy_1+x_1y=2(n-\lambda\cos C) & \ldots\ldots(6). \end{array}$$

From (2), (3), (4), $\quad 2lyz = y^2 + 2yz \sin A + z^2 = P$, say;

" (3), (5), (1), $\quad 2mzx = z^2 + 2zx \cos B + x^2 = Q$, say;

" (1), (2), (6), $\quad 2nxy = x^2 + 2xy \cos C + y^2 = R$, say;

hence we have $\qquad l : m : n = Px : Qy : Rz.$

And as any fourth condition gives a homogeneous equation in (l, m, n), we may easily find the locus of the foci in this case by substituting for these quantities (Px, Qy, Rz).

2. Taking as triangle of reference that formed by the three points, the equation of each conic of the system is of the form $lyz + mzx + nxy = 0$, and the tangential equation is $l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta = 0$ (or Σ); and if $\Sigma - \lambda\Omega \equiv (\alpha x + \beta y + \gamma z)(\alpha x_1 + \beta y_1 + \gamma z_1)$, as before we have

$$\begin{array}{ll|l} xx_1 = l^2 - \lambda \ldots\ldots (1), & & yz_1 + y_1z = 2(\lambda \cos A - mn) \ldots (4), \\ yy_1 = m^2 - \lambda \ldots\ldots (2), & & zx_1 + z_1x = 2(\lambda \cos B - nl) \ldots (5), \\ zz_1 = n^2 - \lambda \ldots\ldots (3), & & xy_1 + x_1y = 2(\lambda \cos C - lm) \ldots (6). \end{array}$$

From (2), (3), (4); (3), (5), (1); (1), (2), (6); we have

$$(ny + mz)^2 = \lambda P, \quad (lz + nx)^2 = \lambda Q, \quad (mx + ly)^2 = \lambda R;$$

hence

$$l : m : n = x(-xP^{\frac{1}{2}} + yQ^{\frac{1}{2}} + zR^{\frac{1}{2}}) : y(xP^{\frac{1}{2}} - yQ^{\frac{1}{2}} + zR^{\frac{1}{2}}) : z(xP^{\frac{1}{2}} + yQ^{\frac{1}{2}} - zR^{\frac{1}{2}}).$$

And as any fourth condition gives a homogeneous equation in (l, m, n), we can eliminate $l : m : n$ by the above proportion, and determine the locus of the foci in this case.

3. Taking as triangle of reference the self-conjugate triangle, the equation of each conic of this system is of the form $ax^2 + by^2 + cz^2 = 0$, and the tangential equation is $bc\alpha^2 + ca\beta^2 + ab\gamma^2 = 0$ or $A\alpha^2 + B\beta^2 + C\gamma^2 \equiv \Sigma$; and $\Sigma - \lambda\Omega \equiv (\alpha x + \beta y + \gamma z)(\alpha x_1 + \beta y_1 + \gamma z_1)$ gives

$$\begin{array}{ll|l} xx_1 = A - \lambda \ldots\ldots\ldots (1), & & yz_1 + y_1z = 2\lambda \cos A \ldots\ldots\ldots (4), \\ yy_1 = B - \lambda \ldots\ldots\ldots (2), & & zx_1 + z_1x = 2\lambda \cos B \ldots\ldots\ldots (5), \\ zz_1 = C - \lambda \ldots\ldots\ldots (3), & & xy_1 + x_1y = 2\lambda \cos C \ldots\ldots\ldots (6). \end{array}$$

From (2), (3), (4); (3), (1), (5); (1), (2), (6); we have

$$Cy^2 + Bz^2 = \lambda P, \quad Az^2 + Cx^2 = \lambda Q, \quad Bx^2 + Ay^2 = \lambda R;$$

hence

$$A : B : C = x^2(-x^2P + y^2Q + z^2R) : y^2(x^2P - y^2Q + z^2R) : z^2(x^2P + y^2Q - z^2R)$$

or $\qquad bc : ca : ab = xyz + x(-x \cos A + y \cos B + z \cos C)$

$$: xyz + y(x \cos A - y \cos B + z \cos C) : xyz + z(x \cos A + y \cos B - z \cos C),$$

substituting for A, B, C and P, Q, R their values. We can thus find $a : b : c$ easily, since $bc : ca : ab = a^{-1} : b^{-1} : c^{-1}$.

And as any fourth condition gives a homogeneous equation in (a, b, c), we can easily eliminate these quantities by the above proportion, and find the locus of the foci in this case.

1749. (Proposed by the Rev. W. Roberts, M.A.)—AB is a diameter of a circle, O its centre, and P a point in OA, such that $OP = \frac{1}{3}OA$. Through P

any straight line QPR is drawn, meeting the circumference in Q, R; and the straight line passing through Q, O is produced to meet that through B, R in S.

If $\angle OSB = \psi$, and $\angle OBS = \phi$, prove that $\dfrac{1-\sin\psi}{1+\sin\psi} = \left(\dfrac{1-\sin\phi}{1+\sin\phi}\right)^3$.

Solution by Dr. Booth, F.R.S.

The following solution of this question will illustrate a new method of analysis, which I have called *Parabolic Trigonometry*, of which an outline was given in a paper read before the British Association at Cheltenham in 1856, and published among the *Reports* of that year. A reference to the same theory will be found in the *Philosophical Transactions* for 1852, p. 386.

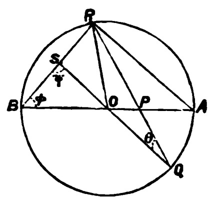
The formula in the Question may be reduced to

$$\sec\psi + \tan\psi = (\sec\phi + \tan\phi)^3;$$

and this latter expression, as I have shown in the paper above referred to, may be put in the form

$$(\sec\phi + \tan\phi)^3 = \sec(\phi \perp \phi \perp \phi) + \tan(\phi \perp \phi \perp \phi),$$

in which $\perp$ and $\top$ denote a relation which I have termed parabolic or logarithmic *plus* and *minus*.



Now, drawing the annexed figure according to the construction given, let $PQ = a$, $OQ = r$, and $\angle OQP = \theta$; then we have

$$2\theta = \psi - \phi, \quad 3a\sin\theta = r\sin(\psi + \phi),$$
$$9a^2 = 10r^2 + 6r^2\cos(\psi + \phi);$$

hence, eliminating θ, a, r, we have

$$2(\sec\phi\sec\psi - \tan\psi\tan\phi)^2$$
$$-(\sec\phi\sec\psi + \tan\phi\tan\psi) = 1.$$

Subtracting $(\sec^2\psi - \tan^2\psi)(\sec^2\phi - \tan^2\phi) = 1$ from the preceding equation, we have

$$(\sec^2\psi + \tan^2\psi)(\sec^2\phi + \tan^2\phi) - 4\sec\psi\tan\psi\sec\phi\tan\phi$$
$$= \sec\phi\sec\psi + \tan\psi\tan\phi.$$

Now, in the paper referred to, it is shown that

$$\sec^2\psi + \tan^2\psi = \sec(\psi \perp \psi), \text{ \&c.}; \quad 2\sec\psi\tan\psi = \tan(\psi \perp \psi), \text{ \&c.};$$
$$\sec\psi\sec\phi + \tan\phi\tan\psi = \sec(\psi \perp \phi), \text{ \&c.};$$

hence, substituting these values in the preceding equation, it becomes

$$\sec(\psi \perp \psi)\sec(\phi \perp \phi) - \tan(\psi \perp \psi)\tan(\phi \perp \phi) = \sec(\psi \perp \phi).$$

But this formula may be written

$$\sec(\psi \perp \psi \top \phi \top \phi) = \sec(\psi \perp \phi), \quad \text{or} \quad \psi \perp \psi \top \phi \top \phi = \psi \perp \phi;$$

or, transposing and changing $\top$ into $\perp$,

$$\psi = \phi \perp \phi \perp \phi, \quad \text{or} \quad \sec\psi + \tan\psi = (\sec\phi + \tan\phi)^3.$$

The following general relation exists in Parabolic Trigonometry; viz.,

$$\sec(\phi \perp \phi \perp \ldots \text{ to } n \text{ angles}) + \tan(\phi \perp \phi \perp \ldots \text{ to } n \text{ angles})$$
$$= (\sec\phi + \tan\phi)^n,$$

just as in Circular Trigonometry, if $i = \sqrt{(-1)}$, we have

$$\cos n\phi + i \sin n\phi = (\cos\phi + i \sin\phi)^n.$$

Thus De Moivre's celebrated theorem on the relation of circular arcs has its dual in the real relations of parabolic arcs.

The curve of which $r = a(\sec\phi \pm \tan\phi)$ is the polar equation, I have called the *Logocyclic Curve*. One of its most important properties is that, if we represent all numbers from 0 to ∞ in succession by the radii of this curve, the conjugate parabola to it will represent by its arcs the logarithms of those numbers.

[NOTE.—The theorem may be otherwise proved as follows:—

$$2 = \frac{PB}{PA} = \frac{PB}{PR} \cdot \frac{PR}{PA} = \frac{\sin\frac{1}{2}(\phi+\psi)}{\sin\phi} \cdot \frac{\cos\phi}{\cos\frac{1}{2}(\phi+\psi)};$$

therefore $$\tan\tfrac{1}{2}(\phi+\psi) = 2\tan\phi \quad\ldots\ldots\ldots\ldots\ldots\ldots (\alpha);$$

whence we obtain, successively,

$$\tan\tfrac{1}{2}\psi = \frac{\tan^3\frac{1}{2}\phi + 3\tan\frac{1}{2}\phi}{3\tan^2\frac{1}{2}\phi + 1}; \quad \frac{1-\tan\frac{1}{2}\psi}{1+\tan\frac{1}{2}\psi} = \left(\frac{1-\tan\frac{1}{2}\phi}{1+\tan\frac{1}{2}\phi}\right)^3;$$

$$\frac{1-\sin\psi}{1+\sin\psi} = \left(\frac{1-\sin\phi}{1+\sin\phi}\right)^3.$$

Moreover, putting $OS = \rho$, $\angle AOS = \chi$, we have

$$\frac{r}{\rho} = \frac{\sin\psi}{\sin\phi} = \frac{\sin(\chi-\phi)}{\sin\phi} = \sin\chi\cot\phi - \cos\chi \quad\ldots\ldots\ldots (\beta).$$

But (α) is equivalent to $\cot\phi = 2\cot\frac{1}{2}\chi$, hence (β) becomes $\rho(1+\frac{1}{2}\cos\chi) = \frac{1}{2}r$, which shows that the *locus of* S is an ellipse whose eccentricity is $\frac{1}{2}$, major axis BP, and focus O.—EDITOR.]

ON RADIAL CURVES.

By R. TUCKER, M.A.

66. In continuation of previous Articles on Radials, we proceed to prove the following equation which we have stated elsewhere. Adopting the notation previously used, and denoting by p' the perpendicular from the Radial pole on the tangent at any point, we have

$$p' = \rho\sin\phi' = -\left(\frac{ds}{d\phi}\right)^2 \div \frac{ds''}{d\phi};$$

hence $$\frac{dp'}{d\phi} = \left\{-2\frac{ds}{d\phi}\cdot\frac{d^2s}{d\phi^2}\cdot\frac{ds''}{d\phi} + \frac{d^2s''}{d\phi^2}\cdot\left(\frac{ds}{d\phi}\right)^2\right\} \div \left(\frac{ds''}{d\phi}\right)^2;$$

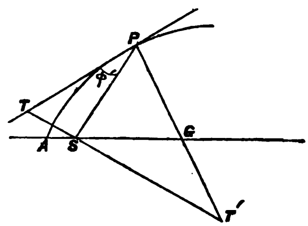
and $$R_1 = \frac{d^2s''}{d\phi_1^2} = \frac{d^2s''}{d\phi^2}\cdot\left(\frac{d\phi}{d\phi_1}\right)^2; \quad \therefore \frac{d^2s''}{d\phi^2} = R_1 \cdot \frac{\rho^2 + R^2}{\rho_1^2};$$

whence $$\frac{dp'}{d\phi} = -\frac{\rho}{\rho_1^2}\left\{\frac{2R\rho_1^2 - \rho R_1\sqrt{(\rho^2+R^2)}}{\sqrt{(\rho^2+R^2)}}\right\};$$

but $\quad \rho_1 = \rho \frac{d\rho}{dp'} = R\rho_1^2 \sqrt{(\rho^2 + R^2)} \div \left\{2R\rho_1^2 - \rho R_1 \sqrt{(\rho^2 + R^2)}\right\}$;

hence
$$\left.\begin{aligned} 2R\rho_1^2 - R\rho_1\sqrt{(\rho^2 + R^2)} &= \rho R_1 \sqrt{(\rho^2 + R^2)} \\ \text{or, by } (\xi), \quad R\rho_1 \left\{2\frac{d\phi}{d\phi_1} - 1\right\} &= \rho R_1 \end{aligned}\right\} \dots\dots\dots\dots\dots (\tau).$$

67. Let PT be the tangent at any point P of the radial AP (pole S), and TST′ a perpendicular to SP meeting the normal PG in T′; then, by (δ) and (ϵ), we have polar subnormal = ST′ = $\rho \cot \phi'$ = radius of curvature at corresponding point of evolute of primitive; and the direction of ST′ is clearly parallel to the direction of the same radius of curvature; hence we readily see that the locus of T′ is the radial of the evolute of the primitive. This furnishes an easy method for finding the radial of the evolute of a curve when the radial of the primitive is known.



68. Now, since if a curve and its evolute are similar their radials must be similar, we see that the problem of Art. 24 is identical with that of finding when the locus of T′ is similar to that of P. We thus easily arrive at the functional equation and result of Art. 24.

69. If we suppose SP, ST′ corresponding lines, we have $\frac{ST'}{SP}$ = a constant = $\cot \phi'$, which is the property of an equiangular spiral. (See above, Art. 25.)

70. If T″, T‴, ... T_n be a series of points (lying alternately on SP, ST′) obtained as in Art. 67, we see that the locus of T_n is the radial of the nth evolute of the primitive of AP.

71. When the locus of T′ is the inverse of that of P, and SP, ST′ corresponding lines, since in this case

$$k^2 = SP \, . \, ST' = r'^2 \cot \phi' = r' \frac{dr'}{d\theta'},$$

we have $r'^2 = 2k^2\theta'$ for the locus of P.

72. We will now show that a somewhat more general solution may be found for equation (A) in Art. 24. The case we propose to discuss is that of $n = 1$. If ϵ be the base of Napierian logarithms, our equations will then be

$$\frac{2x}{c} = \int 2\epsilon^{m\theta} \cos^2 \theta \, d\theta = \frac{\epsilon^{m\theta}}{m}\left\{1 + \frac{m^2 \cos 2\theta + 2m \sin 2\theta}{m^2 + 4}\right\} + c_1,$$

$$\frac{2y}{c} = \int \epsilon^{m\theta} \sin 2\theta \, d\theta = \epsilon^{m\theta}\left\{\frac{m \sin 2\theta - 2 \cos 2\theta}{m^2 + 4}\right\} + c_2;$$

and if now we suppose x and y to vanish with θ, we have

$$c_1 = -\frac{2(m^2 + 2)}{m(m^2 + 4)}, \quad c_2 = \frac{2}{m^2 + 4};$$

hence, re-arranging, we get

$$\epsilon^{m\theta}(m^2\cos 2\theta + 2m\sin 2\theta + m^2 + 4) = \frac{2x}{c}m(m^2+4) + 2(m^2+2) = \mathrm{X},$$

$$\epsilon^{m\theta}(m^2\sin 2\theta - 2m\cos 2\theta) = \frac{2y}{c}m(m^2+4) - 2m = \mathrm{Y};$$

squaring and adding, we have

$$\mathrm{X}^2 + \mathrm{Y}^2 = 2(m^2+4)\epsilon^{2m\theta}(m^2 + 2 + m^2\cos 2\theta + 2m\sin 2\theta),$$

and $\qquad \mathrm{X} = \epsilon^{m\theta}(m^2\cos 2\theta + 2m\sin 2\theta + m^2 + 4)$;

whence we obtain

$$2\epsilon^{2m\theta} - \mathrm{X}\epsilon^{m\theta} + \frac{\mathrm{X}^2+\mathrm{Y}^2}{2(m^2+4)} = 0,$$

or $\qquad \epsilon^{m\theta} = \left\{\mathrm{X}\sqrt{(m^2+4)} \pm \sqrt{(m^2\mathrm{X}^2 - 4\mathrm{Y}^2)}\right\} \div 4\sqrt{(m^2+4)} = t^m$;

therefore $\qquad \theta = \log t,$

and $\qquad \mathrm{Y} = t^m . \left\{m^2\sin(\log t^2) - 2m\cos(\log t^2)\right\},$

the equation to a class of curves whose evolutes are similar to themselves. If we make $m=0$, we get (as we ought) the cycloid as a particular curve of the class.

73. We next proceed to find what curves have for radials the conic sections, the radial pole being the focus of the conic.

We take the equation (e being the eccentricity)

$$r = \frac{c}{1+e\cos\theta}, \quad \text{hence} \quad \frac{dp}{(1+p^2)\left\{\sqrt{(1+p^2)}+ep\right\}} = \frac{dx}{c} = \frac{\cos z\, dz}{1+e\sin z}$$

(if $p = \tan z$); integrating, $\frac{ex}{c} = \log(1+e\sin z)$,

or $\qquad \epsilon^{\frac{ex}{c}} = 1 + e\frac{p}{\sqrt{(1+p^2)}}, \quad$ and $\quad \frac{dy}{dx} = p = \frac{\epsilon^{\frac{ex}{c}} - 1}{\sqrt{\left\{e^2 - (\epsilon^{\frac{ex}{c}} - 1)^2\right\}}}$;

whence, assuming $\epsilon^{\frac{ex}{c}} - 1 = e\cos\phi$, we get

$$\frac{e}{c}dy = -d\phi + \frac{d\phi}{1+e\cos\phi};$$

therefore $\quad ey = \frac{c}{\sqrt{(e^2-1)}}\log\frac{\sqrt{(e-1)}\tan\frac{1}{2}\phi + \sqrt{(e+1)}}{\sqrt{(e-1)}\tan\frac{1}{2}\phi - \sqrt{(e+1)}} - c\phi \ldots\ldots (e>1),$

also $\quad ey = \frac{2c}{\sqrt{(1-e^2)}}\tan^{-1}\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{1}{2}\phi - c\phi \ldots\ldots\ldots (e<1),$

and $\quad y = c\tan\frac{1}{2}\phi - c\phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (e=1),$

where ϕ has the value above assigned.

74. Next, when the radial pole is the centre of the conic. We must discuss the curves separately, and first we take the ellipse, whose equation is

$$r^2 = \frac{b^2}{1 - e^2\cos^2\theta};$$

$$\therefore \frac{dx}{b} = \frac{dp}{(1+p^2)\sqrt{\left\{1+p^2(1-e^2)\right\}}} = \frac{\cos z\, dz}{\sqrt{(1-e^2\sin^2 z)}} \text{ (if } p = \tan z);$$

$$\therefore\ \frac{x}{b} = \sin^{-1}(e \sin z), \quad \text{and} \quad \sin\frac{x}{b} = e\sin z = e\,\frac{p}{\sqrt{(1+p^2)}};$$

therefore
$$\frac{dy}{dx} = p = \frac{-bd\left(\cos\frac{x}{b}\right)}{\sqrt{\left\{\cos^2\frac{x}{b} - (1-e^2)\right\}}};$$

whence
$$\epsilon^{-\frac{y}{b}} = \cos\frac{x}{b} + \sqrt{\left\{\cos^2\frac{x}{b} - (1-e^2)\right\}};$$

and
$$\frac{b^2}{a^2}\epsilon^{\frac{y}{b}} = \cos\frac{x}{b} - \sqrt{\left\{\cos^2\frac{x}{b} - (1-e^2)\right\}};$$

therefore
$$\frac{b^2}{a^2}\epsilon^{\frac{y}{b}} + \epsilon^{-\frac{y}{b}} = 2\cos\frac{x}{b}$$

is the equation to the primitive of the ellipse referred to its centre.

75. Similarly for the hyperbola we have

$$r^2 = \frac{b^2}{e^2\cos^2\theta - 1}, \quad \text{and} \quad \frac{dx}{b} = \frac{\cos z\, dz}{\sqrt{(e^2\sin^2 z - 1)}};$$

therefore
$$\frac{ex}{b} = \log\left\{e\sin z + \sqrt{(e^2\sin^2 z - 1)}\right\};$$

hence
$$\epsilon^{\frac{ex}{b}} + \epsilon^{-\frac{ex}{b}} = 2e\sin z = 2e\,\frac{p}{\sqrt{(1+p^2)}};$$

therefore, if $\epsilon^{\frac{ex}{b}} + \epsilon^{-\frac{ex}{b}} = z'$, we have

$$dy = \frac{bz'\,dz'}{e\sqrt{(4e^2 - z'^2)}\sqrt{(z'^2-4)}} = \frac{bd\,(z')^2}{2e\sqrt{\left\{4(e^2+1)z'^2 - z'^4 - 16e^2\right\}}},$$

and
$$\frac{2ey}{b} = \sin^{-1}\left\{\frac{2z'^2 - 4(e^2+1)}{4(e^2-1)}\right\},$$

or
$$2(e^2-1)\sin\frac{2ey}{b} = \epsilon^{\frac{2ex}{b}} + \epsilon^{-\frac{2ex}{b}} - 2e^2,$$

which reduces, in the case of the rectangular hyperbola, to

$$2\cos\left(\frac{\pi}{4} - \frac{y\sqrt{2}}{b}\right) = \epsilon^{\frac{x\sqrt{2}}{b}} - \epsilon^{-\frac{x\sqrt{2}}{b}}.$$

76. From equation (ι) we have

$$\frac{ds''}{dx} = \frac{(1+p^2)^{\frac{1}{2}}\left\{q^4 + [3pq^2 - r(1+p^2)]^2\right\}^{\frac{1}{2}}}{q^2} \ldots\ldots\ldots\ldots\ldots (v),$$

an equation connecting the arc of a radial with the abscissa of the corresponding point on the primitive.

77. We may apply (v) to the case considered in Art. 55. Since the curve and radial are similar, if s, s'' are corresponding arcs, we have,

since
$$\sqrt{(1+p^2)} = \frac{ds}{dx}, \quad k^2q^4 = q^4 + [3pq^2 - r(1+p^2)]^2;$$

hence $$bq^3 = 3pq^2 - r(1+p^2);$$

therefore $$\frac{r}{q} = \frac{(3p-b)\,q}{1+p^2}, \quad \text{and} \quad b\tan^{-1}p = \log\frac{(1+p^2)^{\frac{3}{2}}}{q};$$

therefore $r' = c\epsilon^{-b\theta'}$ is the equation to the radial, or the curve required, by Art. 17, is the equiangular spiral.

78. From our previous equations we can readily find the locus of a point whose distance from a curve equals the radius of curvature at the corresponding point of that curve. There are two curves fulfilling the conditions, the evolute and the locus we are about to indicate.

Let X′, Y′ be the coordinates of the point; then

$$X' = 2x - \alpha = x + x', \quad Y' = 2y - \beta = y - y';$$

therefore $$dY' = 2dy - d\beta = dy - dy', \quad dX' = 2dx - d\alpha = dx + dx';$$

and $$(dS')^2 = 4(ds)^2 + (ds')^2 = 3(ds)^2 + (ds'')^2;$$

$$\frac{dY'}{dX'} = \frac{x'y' - p'(2x'^2 + y'^2)}{x'^2 + 2y'^2 - p'x'y'}, \text{ by } (\theta), \text{ or } = \frac{-pq^2 + r(1+p^2)}{q^2(2+3p^2) - rp(1+p^2)} \text{ by } (\iota),$$

$$\frac{d^2Y'}{dX'^2} = \frac{q^3\left\{-2q^4 - 2pq^2r(1+p^2) + (1+p^2)^2(2qs - 3r^2)\right\}}{\left\{q^2(2+3p^2) - rp(1+p^2)\right\}^2};$$

$$P' = \frac{(1+p^2)^{\frac{3}{2}}}{q^3}\cdot\frac{\left\{4q^4 + [3pq^2 - r(1+p^2)]^2\right\}^{\frac{3}{2}}}{(1+p^2)^2(2qs - 3r^2) - 2pq^2r(1+p^2) - 2q^4}.$$

79. By elementary geometry we have

$$X'^2 + Y'^2 + \alpha^2 + \beta^2 = 2(y^2 + x^2) + 2(y'^2 + x'^2).$$

80. From our equations (Art. 78) we can show that the circle is the only curve for which this locus is parallel to original curve; also that for the catenary the locus is the axis of X, as we can infer from a known property of the curve.

81. Since $\dfrac{dy}{dx} = \cot\theta'$ $\therefore$ $\dfrac{d^2y}{dx^2} = -\operatorname{cosec}^2\theta'\dfrac{d\theta'}{dx} = \dfrac{1}{r'\sin^3\theta'}$ by (β);

hence $$\frac{dx}{d\theta'} = -r'\sin\theta' = -Y, \quad \text{and} \quad \frac{dy}{d\theta'} = -r'\cos\theta' = -X;$$

therefore $$\frac{d^3y}{dx^3} = \frac{\sin\theta'\dfrac{dr'}{d\theta'} + 3r'\cos\theta'}{r'^3\sin^5\theta'},$$

and similarly for higher differential coefficients.

82. Let v, v', v'' be the velocities of particles at corresponding points of the primitive, its evolute, and its radial, then, employing the usual notation, we have

$$v''^2 = \frac{h''^2}{p'^2} = \frac{h''^2(ds'')^2}{r'^2(ds)^2} = \frac{h''^2}{\rho^4}(\rho^2 + R^2) \ \ldots\ldots\ldots\ldots \text{(i)}.$$

Again, $$\frac{F''}{v''^2} = \rho\left(\frac{d\theta'}{ds''}\right)^2 - \frac{d^2\rho}{ds''^2},$$

and $$\frac{F''(\rho^2 + R^2)}{v''^2} = \rho\left(\frac{d\theta'}{d\phi}\right)^2 - \frac{d^2\rho}{d\phi^2} = \rho - \frac{dR}{d\phi};$$

hence $$F'' = \frac{h''^2}{\rho^4}\left(\rho - \frac{dR}{d\phi}\right) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

Or, $\dfrac{F''}{v'^2}=\dfrac{-ds\,.\,d\theta'}{ds''^2}-\dfrac{d^2\rho}{ds''^2}=\dfrac{1}{\rho}\left(\dfrac{ds}{ds''}\right)^2-\dfrac{d^2\rho}{ds''^2}$

$$=\frac{1}{\rho}-\frac{1}{\rho}\left(\frac{d\rho}{ds''}\right)^2-\frac{d^2\rho}{ds''^2}\ \ldots\ldots\ldots\ldots \text{(iii.)}$$

83. From equation (μ) we see that, if the times of describing the three arcs be the same, then $v''^2=v^2+v'^2$.

84. We may observe that by means of Art. 57 we may conversely, when we can solve the equations, find the primitive corresponding to a given radial.

85. From our definition it readily follows that the sum of the perimeters of all the circles of curvature of the primitive $=2\pi\Sigma(r)$, and the sum of the areas of the same circles $=\frac{1}{2}\pi\Sigma(r^2)$, the summation being taken between the proper limits.

86. We venture to call the reader's attention to two points upon which we are disposed to think some light may be thrown by the preceding discussion, viz., properties of elliptic arcs and differential equations. We have not arrived at any satisfactory results in these two directions, but leave the examination to more competent hands.

Application of Radials to Steiner's Envelope.

1. The Envelope in Quest. 1679 has for some time engaged my attention, and I had obtained some interesting properties, besides Quest. 1649, before I was aware of its having been discussed elsewhere. My present object is more particularly to give such results as readily follow from the application of my method of *Radials* to the Envelope.

2. Reference is made throughout to the Abridgement of Steiner's Paper (*Reprint*, Vol. III., pp. 97—100) and to my articles in the *Reprint*, Vol. I., pp. 16—19, and Vol. II., pp. 27—38.

3. Take for axes a line through S parallel to AB and $S\gamma$, and let $\angle\,ASp=\theta$, then the equation to G will be

$$Y-R\cos C=\cot(B+\tfrac{1}{2}\theta)\,\left\{X-R\sin(C-\theta)\right\}\ldots\ldots\ldots\text{(i.)};$$

hence, by (α) and (β) of *Radials*, we have

$$\cot\theta'=\frac{dy}{dx}=\cot(B+\tfrac{1}{2}\theta),$$

$$(r'\sin^3\theta')^{-1}=\frac{d^2y}{dx^2}=\left\{4R\sin^3\theta'\sin(3\theta'-2B-C)\right\}^{-1};$$

hence $r'=4R\sin(3\theta'-2B-C)$ is the radial of the envelope of G; or, turning the axes through a right angle, $r'=4R\cos(3\theta'+A-B)$; but this, by Art. 17, and Todhunter's *Integral Calculus*, § 112, is the radial to a tricusp, a vertex of the curve lying on a line inclined to $S\gamma$ at an angle $=-\frac{1}{3}(A-B)$; and its intrinsic equation referred to a vertex is $s=\frac{4}{3}R\sin 3\theta$, whence we readily derive (9) the whole length of the curve $=8R=16r$.

4. If ρ, ρ' be the radii of curvature at t, t', and s_1, s_2 the corresponding arcs measured from a vertex, we have, from (5),

$$st=\tfrac{1}{2}\rho'=\tfrac{3}{2}s_1,\quad st'=\tfrac{1}{2}\rho=\tfrac{3}{2}s_2,\quad \rho+\rho'=3\,(s_1+s_2).$$

Analogous properties may be easily established, by means of Art. 17, for any epicycloid or hypocycloid.

5. By means of (3), we see that (m_2) touches the curve, and that at its vertices; hence, by Art. 3 above, the cuspidal tangent through m is inclined to $S\gamma$ at the angle $\frac{1}{3}(A-B)$, which gives us the point w, as in (8), for a vertex.

6. The result of Quest. 1649, which I have proved geometrically on p. 59 of Vol. II. of the *Reprint*, may be proved by combining the equation to G′ with (i).

7. Since the coordinates of p are R sin (C − θ), R cos (C − θ), the equation to a perpendicular from P on G is

$$Y-R\cos(C-\theta) = -\tan\left(B+\tfrac{1}{3}\theta\right)\left\{X-R\sin(C-\theta)\right\} \quad \ldots\ldots \text{(ii.)},$$

the radial for the envelope of which is $r' = 8R\sin(3\theta'+A-B)$, or turned through a right angle $r' = 8R\cos(3\theta'+A-B)$, that is, a tricusp G'_3 similar to G_3 having its cuspidal tangents parallel to those of G_3; hence the envelope of *the axes of a system of parabolas which touch the sides of a triangle is a tricusp* having its parameter twice that of G_3.

8. Now, if we proceed to find the equation to G″, the foot-perpendicular line for the variable triangle ABp with regard to the point C, we see that i tis

$$Y-R\cos C = -\cot\left(B+\tfrac{1}{3}\theta\right)\left\{X-R\sin(A-B)\right\} \quad \ldots\ldots \text{(iii.)};$$

hence G and G″ are supplementally inclined to AB and intersect upon (m_2) in the point μ. By Quest. 1431 (p. 35, Vol. I. of the *Reprint*), the tangents at the vertices of the four parabolas, which have three points on a circle for intersection of tangents and the fourth for a focus, pass through a point (μ); and the locus of μ, by the above, is for three fixed points the nine-point circle of the triangle formed by them.

9. Also we see that the nine-point circles of the four triangles pass through the same point μ, and have, by Quest. 1431,

$$(\mu A)^2 + (\mu B)^2 + (\mu C)^2 + (\mu p)^2 = D^2,$$

as we may very readily prove, by the above property, if we draw the figure.

10. The perpendiculars from p and p' on G, G′, respectively, manifestly intersect at right angles on (S); hence (S) is situated with respect to G'_3 in the same way that (m_2) is to G_3, and a little examination shows us that the vertices of G'_3 are situated on lines through S parallel to mw, &c., towards the cusps of G_3.

11. Similar properties, of course, obtain for G'_3 as for G_3, the cusps lying on a circle, centre S, radius 3R, &c.

12. If Aa, Bb, Cc, touch G_3 in L_1, L_2, L_3, then

$$DL_1 \sin A = DL_2 \sin B = DL_3 \sin C = 2R \sin A \sin B \sin C.$$

N.B.—The subject of Radial Curves first occupied the writer's attention in February, 1863, and he was induced to work further at these curves by the persuasion of his able friend the Editor, who from the outset expressed his belief in the utility of the method.

1711. (Proposed by Professor SYLVESTER.)—To a uniform beam supported at its extremities is suspended a weight equal to that of the beam; show that, if the point of suspension is nearer to the centre than to an end of the beam, the moment of strain will be greatest at that point, but otherwise greatest at a point at the same distance from the centre, as the point of suspension is from the end of the beam.

Solution by the REV. R. TOWNSEND, M.A.

The answer to this question follows immediately from the following principles :—

I. *A horizontal beam* AB, *supported at both extremities* A *and* B, *is loaded with a weight* E *at an arbitrary point* P ; *the bending moment, due to the action of* E, *at any point* X *of its length, is* $\frac{E}{AB}\,.\,BP\,.\,AX$, *or* $\frac{E}{AB}\,.\,AP\,.\,BX$, *according as* X *is situated between* A *and* P, *or between* B *and* P.

For, the beam AB being held in constrained equilibrium by the weight E at the point P and by the two reactions $E\,.\,\frac{BP}{AB}$ and $E\,.\,\frac{AP}{AB}$ at its two extremities A and B, the entire bending moment at any point X of its length is consequently that arising from the reaction at the extremity A or B between which and the point P it lies.

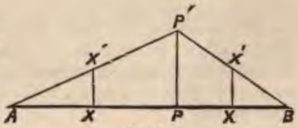


Fig 1.

COR. 1.—It follows from the above, that, for different positions of X along the beam, the bending moment, which is of course evanescent at each extremity A and B, increases uniformily from each to the point P, where it is the maximum, and has for value $\frac{E}{AB}\,.\,AP\,.\,BP$.

COR. 2.—The following construction for representing geometrically the magnitude of the bending moment, at any point X of the beam, follows also from the above. At the point of application P of the load E, draw a perpendicular PP′ to represent on any scale the magnitude of the bending moment at that point, and join AP′ and BP′; then the perpendicular XX′ at any point X, terminated by AP′ or BP′, will represent on the same scale its magnitude at that point.

II. *A horizontal beam* AB, *supported at both extremities* A *and* B, *is loaded with a number of weights* E, F, G, H, *&c., arbitrarily applied at different points* P, Q, R, S, *&c.; the entire bending moment, due to their combined action, at any point* X *of its length, is*

$$\frac{1}{AB}\left\{\Sigma(E\,.\,AP)\,.\,BX + \Sigma(F\,.\,BQ)\,.\,AX\right\};$$

the first summation comprising all the points of application between A *and* X, *and the second all between* B *and* X.

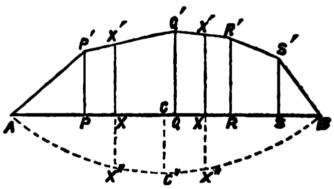


Fig. 2.

This follows at once from Prop. I.; each term of each summation being, by it, the part of the entire moment due to the corresponding weight.

Cor. 1.—It follows from the above, that, during the passage of X along the beam from either extremity A to the other B, the bending moment, which of course vanishes at each extremity, varies uniformly during the passage from each weight to the next; that in the passage over each weight it undergoes an abrupt change; that it always attains its greatest value as some one of them is reached; and that, if it have equal values for any two contiguous weights, it has a constant value equal to each for all positions of X intermediate between them.

Cor. 2.—The following construction for representing geometrically the magnitude of the bending moment, at any point X of the beam, follows also from the above, and shows very clearly the several particulars just enumerated in Cor. 1. At the several points of application P, Q, R, S, &c., of the several loads E, F, G, H, draw the several perpendiculars PP′, QQ′, RR′, SS′, &c., to represent on any common scale the several magnitudes of the bending moment at those points, and join successively AP′, P′Q′, Q′R′, R′S′, &c., and finally S′B; then the perpendicular XX′, drawn from any point X (between P and Q suppose) and terminated by the corresponding connector (P′Q′ in that case), will represent on the same scale its magnitude at that point.

N.B.—If the entire load, in place of being applied in detached portions E, F, G, H, &c., at isolated points P, Q, R, S, &c., of the beam, were distributed continuously along its entire length according to any law, the polygon AP′Q′R′S′ . . . B in the above construction would then be a continuous curve, and the bending moment, which would then vary continuously along its entire length, would be represented at any point X by the ordinate XX′ to that curve. The following is the simplest case of such a distribution.

III. *A horizontal beam* AB, *supported at both extremities* A *and* B, *is loaded with a weight* W *distributed uniformly along its entire length; the entire bending moment due to its action at any point* X *is* $\frac{\frac{1}{2}W}{AB} . AX . BX$; *that is* (*by Prop.* I.) *equal to half of what it would be if the entire load were applied at that point.*

For, in the general formula for the bending moment at X (given for any law of distribution in Prop. II.), the two sums $\Sigma(E . AP)$ and $\Sigma(F . BQ)$, which are then of course integrals, have respectively for values $\frac{W}{AB} . \frac{AX^2}{2}$ and $\frac{W}{AB} . \frac{BX^2}{2}$; therefore the bending moment itself has for value

$$\frac{\frac{1}{2}W}{AB^2} . (AX^2 . BX + BX^2 . AX),$$

which, since $AX + BX = AB$, is manifestly equivalent to the above.

Cor. 1.—It follows from the above, that, as X traverses the length of the beam from one extremity to the other, the bending moment, which, as in

every other case, vanishes of course at each extremity, varies continuously throughout the passage, has equal values at all pairs of points equidistant from its middle point C, and has there its greatest value of which the magnitude is $\frac{1}{8}$W . AB.

Cor. 2.—The following construction for representing geometrically the bending moment at any point X follows also from the above. At the middle point C (Fig. 2) of the beam AB draw the perpendicular CC″, to represent on any scale the magnitude of the bending moment at that point, and with C″ as vertex describe on AB as chord the arc AC″B of the parabola thus determined (see the dotted line in Fig. 2); then the ordinate XX″ to that curve at any point X will represent on the same scale its magnitude at that point.

IV. *A uniform horizontal beam* AB, *supported at both extremities* A *and* B, *is loaded at any point* P *with a weight* W *equal to its own; the entire bending moment due to the combined weights of the load and beam is greatest at the point of application* P *if it be nearer to the centre* C *than to either extremity* A *or* B *of the beam, but otherwise greatest at the point* X, *whose distance* CX *from the centre* C *in the direction of* P *is equal to the distance* AP *of* P *from the nearer extremity* A *of the beam.* (Sylvester's Question.)

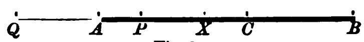


Fig. 3.

For since, for any point X between P and C, where the point of greatest moment must evidently lie, the portions of the bending moment due to the weights of the load and beam are, by Props. I. and III., respectively, $\frac{W}{AB}$. AP. BX and $\frac{\frac{1}{2}W}{AB}$.AX.BX; if Q be the point on AB produced at which QA = 2AP, the sum of the two, that is, the entire moment, is $\frac{\frac{1}{2}W}{AB}$. QX . BX; which is the greatest possible when X, consistently with the restriction of being between P and C, is as near as possible to the middle point of QB.

N.B.—The general problem of which the above is a particular case, viz., "For a uniform beam AB of given weight W, loaded with given weights E, F, G, H, &c., at given points P, Q, R, S., &c., to determine the point (or points) of maximum bending moment under the joint actions of its own and of the appended weights," may be solved readily by aid of the two constructions of Fig. 2 for the bending moments. For, the several perpendiculars PP′, QQ′, RR′, SS′, CC″ being drawn (as explained in Props. II. and III.) to represent on any common scale the several bending moments at P, Q, R, S, &c., due to the combined action of E, F, G, H, &c., at those points, and that at C due to the separate action of W; and the polygon AP′Q′R′S′.... B and parabola AC″B being constructed (as there also explained); every point X, the double ordinate X′X″ at which meets the parabola at a point X″ at which the tangent is parallel to the corresponding side (P′Q′ suppose) of the polygon, is evidently a point at which the bending moment (represented always by that ordinate) is a maximum; and so also, for the same reason, is every point of application (Q suppose) of an appended weight, the tangent at the intersection Q″ of whose ordinate with the curve determines with the corresponding angle P′Q′R′ of the polygon a triangle to which that angle is internal. These two cases evidently correspond respectively to the second and first cases in Professor Sylvester's Question.

Of course, between every two consecutive positions of maximum moment (when more than one exist) a position of minimum moment lies. Every such position corresponds evidently to a point of application (R suppose) of an

appended weight for which the tangent at the corresponding point R″ of the parabola determines with the corresponding angle Q′R′S′ of the polygon a triangle to which the vertically opposite angle is internal; positions of minimum corresponding to those of maximum moment of the first species above noticed evidently not existing from the nature of the case.

1744. (Proposed by W. S. Burnside, B.A.)—It is required to find (x_1, y_1, z_1), functions of (x, y, z), such that we may have identically

$$\frac{x_1^3+y_1^3+z_1^3}{x_1y_1z_1} = \frac{x^3+y^3+z^3}{xyz}.$$

I. *Solution by* Professor Cayley.

The Solution is in fact given in my "Memoir on Curves of the Third Order," *Philosophical Transactions*, vol. 147 (1857), pp. 415—446.

Write $\dfrac{x^3+y^3+z^3}{xyz} = -6l$; then, taking (X, Y, Z) as current coordinates, (x, y, z) are, it is clear, the coordinates of a point on the cubic curve $X^3+Y^3+Z^3+6lXYZ=0$; and if (x_1, y_1, z_1) are the coordinates of any other point on the same cubic curve, then we shall have

$$\frac{x_1^3+y_1^3+z_1^3}{x_1y_1z_1} = -6l = \frac{x^3+y^3+z^3}{xyz},$$

so that (x_1, y_1, z_1) will satisfy the condition in question. Hence, if from a given point (x, y, z) on the cubic curve we obtain by any geometrical construction another point on the curve, the coordinates of this new point will be functions (and, if the construction is such as to lead to a single point only, they will be *rational* functions) of (x, y, z), satisfying the condition in question.

For instance, if the point (x, y, z) be joined with any point (α, β, γ) on the curve, the joining line will again meet the curve in a single point, which may be taken to be the point (x_1, y_1, z_1). This assumes that we know on the cubic curve a point (α, β, γ); but such a point at once presents itself, viz., we may write $(\alpha, \beta, \gamma) = (1, -1, 0)$; which gives only the self-evident solution $(x_1, y_1, z_1) = (y, x, z)$. The point $(1, -1, 0)$ is clearly one of the nine points of inflexion of the cubic curve, and by using these in any manner whatever, viz., joining the point (x, y, z) with any point of inflexion, and then the new point with any other point of inflexion, and so on indefinitely, we obtain in connexion with the given point (x, y, z) seventeen other points on the curve, in all a system of eighteen points: these are

$$\begin{array}{lll|lll}
(x, y, z), & (x, \omega y, \omega^2 z), & (x, \omega^2 y, \omega z) & (x, z, y), & (x, \omega z, \omega^2 y), & (x, \omega^2 z, \omega y) \\
(y, z, x), & (\omega y, \omega^2 z, x), & (\omega^2 y, \omega z, x) & (z, y, x), & (\omega z, \omega^2 y, x), & (\omega^2 z, \omega y, x) \\
(z, x, y), & (\omega^2 z, x, \omega y), & (\omega z, x, \omega^2 y) & (y, x, z), & (\omega^2 y, x, \omega z), & (\omega y, x, \omega^2 z)
\end{array}$$

possessing remarkable geometrical properties; and of course each of the seventeen new points furnishes a (self-evident) solution of the given identity.

But we may take $(\alpha, \beta, \gamma) = (x, y, z)$; the point (x_1, y_1, z_1) is here the point of intersection of the cubic by the tangent at the point (x, y, z); or say it is the "tangential" of the point (x, y, z). The values thus obtained for (x_1, y_1, z_1) are

$$(x_1, y_1, z_1) = \left\{x(y^3-z^3),\ y(z^3-x^3),\ z(x^3-y^3)\right\},$$

which (excluding the above mentioned self-evident solutions) is in fact the most simple solution of the proposed identity. In order to verify that the last mentioned values of (x_1, y_1, z_1) are in fact the coordinates of the tangential of (x, y, z), I observe that this will be the case if only we have

$$(x^2+2lyz)\,x_1 + (y^2+2lzx)\,y_1 + (z^2+2lxy)\,z_1 = 0, \quad x_1^3+y_1^3+z_1^3+6lx_1y_1z_1 = 0,$$

the first of which is obviously satisfied by the values in question; and for the verification of the second equation,

$$\begin{aligned} x_1^3+y_1^3+z_1^3 &= x^3(y^3-z^3)^3+y^3(z^3-x^3)^3+z^3(x^3-y^3)^3 \\ &= -x^9(y^3-z^3)-y^9(z^3-x^3)-z^9(x^3-y^3) \\ &= (x^3+y^3+z^3)(y^3-z^3)(z^3-x^3)(x^3-y^3), \\ x_1y_1z_1 &= xyz(y^3-z^3)(z^3-x^3)(x^3-y^3), \end{aligned}$$

$$\therefore x_1^3+y_1^3+z_1^3+6lx_1y_1z_1 = (x^3+y^3+z^3+6lxyz)(y^3-z^3)(z^3-x^3)(x^3-y^3), = 0$$

if $x^3+y^3+z^3+6lxyz = 0$; the same equations verify at once the identity

$$\frac{x_1^3+x_1^3+z_1^3}{x_1y_1z_1} = \frac{x^3+y^3+z^3}{xyz}.$$

Another solution is as follows: viz., if we take the third intersection with the cubic of the line joining the points (y, x, z) and $\left\{x(y^3-z^3), y(z^3-x^3), z(x^3-y^3)\right\}$, the coordinates of the line in question are

$$\begin{aligned} x_1 : y_1 : z_1 = \ & x^6y^3+y^6z^3+z^6x^3-3x^3y^3z^3 \\ & : x^3y^6+y^3z^6+z^3x^6-3x^3y^3z^3 \\ & : xyz(x^6+y^6+z^6-y^3z^3-z^3x^3-x^3y^3). \end{aligned}$$

According to a very beautiful theorem of Professor Sylvester's in relation to the theory of cubic curves, the coordinates of a point which depends linearly on a given point of the curve are necessarily rational and integral functions *of a square degree* of the coordinates (x, y, z) of the given point; and moreover that (considering as *one* solution those which can be derived from each other by a mere permutation of the coordinates, or change of x into ωx &c.), there is only one solution of a given square degree m^2; the solutions of the degrees 4 and 9 are given above. The tangential of the tangential, or second tangential of the point (x, y, z), gives the solution of the degree 16; joining this second tangential with the original point (x, y, z), we have the solution of the degree 25; and the same solution is also given as the sixth point of intersection with the cubic, of the conic of 5-pointic intersection at the point (x, y, z). See my memoir "On the conic of 5-pointic contact at any point of a plane curve," *Philosophical Transactions*, vol. 149 (1859), pp. 371—400.

Addition to the foregoing Solution. On a system of Eighteen Points on a Cubic Curve.

Considering the cubic curve $x^3+y^3+z^3+6lxyz = 0$, we have the nine points of inflexion, which I represent as follows:—

$$\begin{array}{lll} a = (0, 1, -1), & d = (-1, 0, 1), & g = (1, -1, 0), \\ b = (0, 1, -\omega), & e = (-\omega, 0, 1), & h = (1, -\omega, 0), \\ c = (0, 1, -\omega^2), & f = (-\omega^2, 0, 1), & i = (1, -\omega^2, 0), \end{array}$$

viz., ω being an imaginary cube root of unity, the coordinates of a are $(0, 1, -1)$, those of b, $(0, 1, -\omega)$, &c.

The points of inflexion lie (as is known) by threes on twelve lines; viz., the lines are

abc,	*afh*,	*bfg*,	*cfi*,
adg,	*bdi*,	*cdh*,	*def*,
aei,	*beh*,	*ceg*,	*ghi*.

Consider now a point on the curve, the coordinates whereof are (x, y, z), where of course $x^3+y^3+z^3+6lxyz=0$; this is one of a system of eighteen points on the curve, which may be represented as follows:—

$$\begin{array}{lll} A=(x, y, z), & D=(x, \omega y, \omega^2 z), & G=(x, \omega^2 y, \omega z), \\ B=(y, z, x), & E=(\omega y, \omega^2 z, x), & H=(\omega^2 y, \omega z, x), \\ C=(z, x, y), & F=(\omega^2 z, x, \omega y,) & I=(\omega z, x, \omega^2 y), \\[6pt] J=(x, z, y), & M=(x, \omega z, \omega^2 y), & P=(x, \omega^2 z, \omega y), \\ K=(z, y, x), & N=(\omega z, \omega^2 y, x), & Q=(\omega^2 z, \omega y, x), \\ L=(y, x, z), & O=(\omega^2 y, x, \omega z), & R=(\omega y, x, \omega^2 z), \end{array}$$

viz., the coordinates of A are (x, y, z); those of B are (y, z, x), &c.

The tangent at A meets the curve in a point, "the tangential of A," the coordinates whereof are $x(y^3-z^3)$, $y(z^3-x^3)$, $z(x^3-y^3)$; which point may be called A′. And we have thus the eighteen tangentials

A′, B′, C′, D′, E′, F′, G′, H′, I′, J′, K′, L′, M′, N′, O′, P′, Q′, R′.

The eighteen points A, B, &c., have the following property; viz., the line joining any two of them meets the cubic in a third point, which is either one of the nine points of inflexion, or one of the eighteen tangentials; there are through each point of inflexion 9 such lines, and through each tangential 4 such lines; $(9\times 9)+(18\times 4)=153=\frac{1}{2}(18.17)$, the number of pairs of points AB, AC, &c. The lines through the inflexions are the 81 lines obtained by joining any one of the points (A, B, C, D, E, F, G, H, I) with any one of the points (J, K, L, M, N, O, P, Q, R), as shown in the following Table:—

	A	B	C	D	E	F	G	H	I
J	*a*	*d*	*g*	*c*	*f*	*i*	*b*	*e*	*h*
K	*d*	*g*	*a*	*f*	*i*	*c*	*e*	*h*	*b*
L	*g*	*a*	*d*	*i*	*c*	*f*	*h*	*b*	*e*
M	*c*	*f*	*i*	*b*	*e*	*h*	*a*	*d*	*g*
N	*f*	*i*	*c*	*e*	*h*	*b*	*d*	*g*	*a*
O	*i*	*c*	*f*	*h*	*b*	*e*	*g*	*a*	*d*
P	*b*	*e*	*h*	*a*	*d*	*g*	*c*	*f*	*i*
Q	*e*	*h*	*b*	*d*	*g*	*a*	*f*	*i*	*c*
R	*h*	*b*	*e*	*g*	*a*	*d*	*i*	*c*	*f*

viz., the line AJ passes through a, the line AK through d, &c.; the proof that AJ passes through a depends on the identical equation

$$\begin{vmatrix} x, & y, & z \\ x, & z, & y \\ 0, & 1, & -1 \end{vmatrix} = 0;$$

and the like for the other lines AK, AL, &c.

The lines through the tangentials are the 36 lines obtained by joining any two of the points (A, B, C, D, E, F, G, H, I) and the 36 lines obtained by joining any two of the points (J, K, L, M, N, O, P, Q, R); and these 72 lines pass through the tangentials, as shown by the table

ABC,	BDI,	CEG,	JKL,	KMR,	LNP,
ADG,	BEH,	CFI,	JMP,	KNQ,	LOR,
AEI,	BFG,	DEF,	JNR,	JOP,	MNO,
AFH,	CDH,	GHI,	JOQ,	LMQ,	PQR,

viz., in the triad ABC, BC passes through A′, CA through B′, AB through C′; and the like for the other triads. The proof that BC passes through A depends on the identical equation

$$\begin{vmatrix} y, & z, & x \\ z, & x, & y \\ x(y^3-z^3), & y(z^3-x^3), & z(x^3-y^3) \end{vmatrix} = 0;$$

and the like for the other combinations of points.

If we attend only to the points A, B, C and their tangentials A′, B′, C′; then we have on the cubic three points A, B, C, such that the line joining any two of them passes through the tangential of the third point. And the figure may be constructed by means of the three real points of inflexion a, d, g, as follows, viz., joining these with any point J on the cubic, the lines so obtained respectively meet the cubic in three new points which may be taken for the points A, B, C. Or if one of these points, say A, be given, then joining it with one of the three real inflexions, this line again meets the cubic in the point J, and from it by means of the other two real inflexions we obtain the remaining points B and C; it is clear that, A being given, the construction gives three points, say J, K, L, each of them leading to the same two points B and C.

We may consider the question from a different point of view. Let A, B, C be given points, and let there be given also three lines passing through these three points respectively; through the given points, touching at these points the given lines respectively, describe a cubic; and let the given lines again meet the cubic in the points A′, B′, C′ respectively. The equation of the cubic contains three arbitrary parameters; but when two of these are properly determined, the points A, B, C and their tangentials A′, B′, C′ will be related as in the theorem; viz., the line through any two of the points will pass through the tangential of the third point. The analytical investigation is as follows:—

Let the equations of the three tangents be $x = 0$, $y = 0$, $z = 0$, and suppose that, for the points A, B, C respectively, we have

$$(x = 0,\ y = \lambda z),\ (y = 0,\ z = \mu x),\ (z = 0,\ x = \nu y),$$

then the equation of a cubic touching the three lines at the three points respectively will be

$$(y-\lambda z)^2(\nu^2 By + Cz) + (z-\mu x)^2(\lambda^2 Cz + Ax) + (x-\nu y)^2(\mu^2 Ax + By)$$
$$-\mu^2 Ax^3 - \nu^2 By^3 - \lambda^2 Cz^3 + Kxyz = 0,$$

where A, B, C, K are arbitrary coefficients; but if $A : B : C = \lambda : \mu : \nu$, then the equation is

$$(y-\lambda z)^2 \nu(\mu\nu y+z)+(z-\mu x)^2 \lambda(\nu\lambda z+x)+(x-\nu y)^2 \mu(\lambda\mu x+y)$$
$$-\lambda\mu^2x^3-\mu\nu^2y^3-\nu\lambda^2z^3+Kxyz=0,$$

where

A, A′ are the intersections of $x=0$, by $y-\lambda z=0$, $\mu\nu y+z=0$ respectively,
B, B′ „ „ $y=0$, „ $z-\mu x=0$, $\nu\lambda z+x=0$ „
C, C′ „ „ $z=0$, „ $x-\nu y=0$, $\lambda\mu x+y=0$ „

the equations of BC, CA, AB thus are

$$-\mu x+\mu\nu y+z=0, \quad x-\nu y+\nu\lambda z=0, \quad \lambda\mu x+y-\lambda z=0,$$

which pass through A′, B′, and C′, respectively.

If we consider along with the points A, B, C the points J, K, L, and their respective tangentials, then we have inscribed in the cubic a hexagon ALBJCK which has the following properties, viz., the pairs of opposite sides and the three diagonals pass through the three real inflexions *in lineâ*, viz.,

AL,	JC,	BK,	through g
LB,	CK,	JA,	„ a
BJ,	KA,	CL,	„ d.

This shows that the six points A, B, C, J, K, L are the intersections of the cubic by a conic; and moreover, considering the triangles ABC, JKL formed by the alternate vertices, then in each triangle the sides pass through the tangentials of the opposite vertices respectively.

In what precedes we have in effect found the coordinates (z, x, y) of the third point of intersection with the cubic, of the line joining the points (y, z, x) and $\left\{x(y^3-z^3),\ y(z^3-x^3),\ z(x^3-y^3)\right\}$. The coordinates of the same point may be otherwise found by a direct investigation, as follows:

Write

$$x_2 : y_2 : z_2 = x(y^3-z^3) : y(z^3-x^3) : z(x^3-y^3); \quad x_1 : y_1 : z_1 = y : z : x.$$

If in the equation of the curve we substitute for x, y, z, the values ux_1+vx_2, uy_1+vy_2, uz_1+vz_2, we find

$$u\left\{x_1^2x_2+y_1^2y_2+z_1^2z_2+2l\,(x_2y_1z_1+y_2z_1x_1+z_2x_1y_1)\right\}$$
$$+v\left\{x_1x_2^2+y_1y_2^2+z_1z_2^2+2l\,(x_1y_2z_2+y_1z_2x_2+z_1x_2y_2)\right\}=0,$$

say $uP+vQ=0$; we may therefore write $u=Q$, $v=-P$, and the coordinates of the third point are Qx_1-Px_2, Qy_1-Py_2, Qz_1-Pz_2. Now

$$\begin{aligned}Qx_1-Px_2 &= y_1y_2(x_1y_2-x_2y_1)+z_1z_2(x_1z_2-x_2z_1)+2l\,(x_1^2y_2z_2-x_2^2y_1z_1)\\
&= yz(z^3-x^3)\left\{y^2(z^3-x^3)-zx(y^3-z^3)\right\}\\
&\quad+zx(x^3-y^3)\left\{yz(x^3-y^3)-x^2(y^3-z^3)\right\}\\
&\quad+2l\left\{y^2.\,yz\,.\,(z^3-x^3)(x^3-y^3)-x^2(y^3-z^3)^2zx\right\}.\\
&= (x^3y^6+y^3z^6+z^3x^6-3x^3y^3z^3)z\\
&\quad+xyz(x^6+y^6+z^6-y^3z^3-z^3x^3-x^3y^3)z\\
&\quad-2l\,(x^6y^3+y^6z^3+z^6x^3-3x^3y^3z^3)z;\end{aligned}$$

so that we have $Qx_1 - Px_2 = \Pi z$; and in like manner $Qy_1 - Py_2 = \Pi x$, $Qz_1 - Pz_2 = \Pi y$; and therefore $Qx_1 - Px_2 : Qy_1 - Py_2 : Qz_1 - Pz_2 = z : x : y$, which proves the theorem.

I consider in like manner the following question; viz., if (y, x, z) be joined with the tangential of (x, y, z); to find the third point of intersection. We have here

$$x_2 : y_2 : z_2 = x(y^3 - z^3) : y(z^3 - x^3) : z(x^3 - y^3); \quad x_1 : y_1 : z_1 = y : x : z;$$

and P, Q as before; and the coordinates of the third point are

$$Qx_1 - Px_2 : Qy_1 - Py_2 : Qz_1 - Pz_2;$$

also
$$\begin{aligned} Qx_1 - Px_2 &= xy\,(z^3 - x^3)\,\{y^2(z^3 - x^3) - x^2(y^3 - z^3)\} \\ &\quad + z^2(x^3 - y^3)\,\{yz\,(x^3 - y^3) - zx\,(y^3 - z^3)\} \\ &\quad + 2l\,\{y^3 z\,(z^3 - x^3)(x^3 - y^3) - x^2(y^3 - z^3)^2 zx\} \\ &= x\,\{y^3(z^3 - x^3)^2 - z^3(x^3 - y^3)(y^3 - z^3)\} \\ &\quad + y\,\{z^3(x_3 - y^3)^2 - x^3(y^3 - z^3)(z^3 - x^3)\} \\ &\quad + 2lz\,\{y^3(z^3 - x^3)(x^3 - y^3) - x^3(y^3 - z^3)^2\} \end{aligned}$$

that is $\quad Qx_1 - Px_2 = (x + y - 2lz)(x^3z^6 + y^3x^6 + z^3y^6 - 3x^3y^3z^3);$

similarly $\quad Qy_1 - Py_2 = (x + y - 2lz)(y^3z^3 + y^6z^3 + z^3x^6 - 3x^3y^3z^3);$

also $\quad Qz_1 - Pz_2 = (x + y - 2lz)(x^6 + y^6 + z^6 - y^3z^3 - x^3y^3)\,xyz;$

and we hence have the values

$$\begin{aligned} X : Y : Z = {} & x^6y^3 + y^6z^3 + z^6x^3 - 3x^3y^3z^3 : x^3y^6 + y^3z^6 + z^3x^6 - 3x^3y^3z^3 \\ & : xyz\,(x^6 + y^6 + z^6 - y^3z^3 - z^3x^3 - x^3y^3) \end{aligned}$$

for the coordinates of the point in question.

II. *Solution by the* PROPOSER; Professor CREMONA; S. BILLS; *and others.*

Since $x(y - z) + y(z - x) + z(x - y) = 0$, we have

$$\{x(y - z) + y(z - x) + z(x - y)\}\{(y - z)^2 + (z - x)^2 + (x - y)^2\} =$$
$$\Sigma x\,(y - z)^3 - (y - z)(z - x)(x - y)\ \Sigma\,(y + z - x) = 0;$$

$$\therefore\ x(y - z)^3 + y(z - x)^3 + z(x - y)^3 = (x + y + z)(y - z)(z - x)(x - y).$$

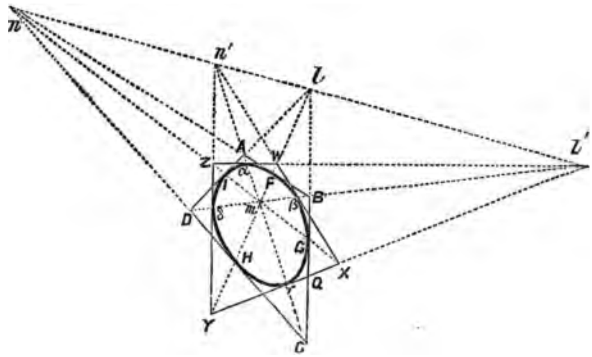
Changing x, y, z into x^3, y^3, z^3, and putting x_1, y_1, z_1 for $x(y^3 - z^3)$, $y(z^3 - x^3)$, $z(x^3 - y^3)$, we have

$$x_1^3 + y_1^3 + z_1^3 = (x^3 + y^3 + z^3)(y^3 - z^3)(z^3 - x^3)(x^3 - y^3),$$

also
$$x_1y_1z_1 = xyz\,(y^3 - z^3)(z^3 - x^3)(x^3 - y^3),$$

therefore
$$\frac{x_1^3 + y_1^3 + z_1^3}{x_1y_1z_1} = \frac{x^3 + y^3 + z^3}{xyz}.$$

1751. (Proposed by Professor CAYLEY.)—Let ABCD be any quadrilateral. Construct, as shown in the figure, the points F, G, H, I: in BC find a point Q such that $\frac{BG}{BC} \cdot \frac{CQ}{GQ} = \frac{1}{\sqrt{2}}$; and complete the construction as shown in the figure. Show that an ellipse may be drawn passing through the eight points F, G, H, I, α, β, γ, δ, and having at these points respectively the tangents shown in the figure.



Solution by PROFESSOR CREMONA.

Conservons la figure de M. Cayley, et désignons, de plus, par des lettres les points (BC, AD) = l, (CA, BD) = m, (AB, CD) = n, (BD, ln) = l', (AC, ln) = n'. On sait, par les propriétés connues du quadrilatère complet (AC, BD, ln), que les systèmes (AB, Fn), (BC, Gl), (CD, Hn), (DA, Il) sont harmoniques; on sait en outre que quatre points pris dans les côtés d'un quadrilatère complet et tels qu' ils forment avec les ternes de sommets le même rapport anharmonique sur chaque côté, sont les points de contact d'une conique inscrite. Donc les droites AB, BC, CD, DA touchent en F, G, H, I une même conique; et pour cette conique le quadrilatère circonscrit est *harmonique*, parceque chaque côté est divisé harmoniquement par les trois autres et par le point de contact. Les points m, n' sont, par rapport à cette conique, les poles des droites ln, BD; donc la polaire de l' est mn' savoir AC; c'est-à-dire que les points α, γ, où la conique est touchée par les tangentes issues du point l', sont collinéaires avec mn'AC. De même les points β, δ où la conique est touchée par les tangentes issues de n' sont sur la droite ml'BD. Ces quatre tangentes issues de l' et n' forment un second quadrilatère circonscrit harmonique; car ex. g. les 4 points (WZ, $l'\alpha$) sont perspectifs aux 4 points (ln, $l'n'$) qui forment un système harmonique.

On peut observer encore que, des propriétés connues du quadrilatère complet (XZ, WY, $l'n'$), pour lequel le triangle diagonal est lmn, il suit évidemment que les droites $\alpha\delta$, $\beta\gamma$ passent par l, et que les droites $\alpha\beta$, $\gamma\delta$ passent par n; de même que l' est l'intersection de FI, GH, et n' est l'intersection de FG, HI.

Pour construire le point Q, duquel dépend le nouveau quadrilatère, calculons le rapport anharmonique (BQGC) = x. Le point Q étant un point double de l'involution (Bl, GC,...), on aura l'égalité (BQGC) = (Ql GC), et par

conséquent (QlGC) $= x$. De cette égalité et de cette autre (BlGC) $= \frac{1}{2}$, qui exprime l'harmonie du système (BC, Gl), on tire par la division (BQGC) $= \frac{1}{2x}$. Mais l'on a (BQGC) $= x$; donc $x^2 = \frac{1}{2}$, ce qui donne les deux points doubles de l'involution, c'est-à-dire les points où BC est coupée par les tangentes issues de l'.

[Professor CAYLEY remarks that if ABCD is the perspective representation of a square, then the ellipse is the perspective representation of the inscribed circle; the theorem gives eight points and the tangent at each of them; and the ellipse may therefore be drawn by hand with an accuracy quite sufficient for practical purposes. The demonstration is immediate, by treating the figure as a perspective representation: the gist of the theorem is the very convenient construction in perspective which it furnishes.]

1695. (Proposed by H. R. GREER, B.A.)—If u_m be a homogeneous function of the degree m in any number of independent variables, prove that

$$\left(x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dx}\dots\right)\log u_m = m, \text{ identically};$$

$$\left(x^2\frac{d^2}{dx^2} + y^2\frac{d^2}{dy^2}\,\&\text{c.} + 2xy\frac{d}{dx}\cdot\frac{d}{dy}\,\&\text{c.}\right)\log u_m = -m;$$

and continue the theorem.

Solution by F. D. THOMSON, M.A.

Let $f(x, y, z, \dots) \equiv \log u_m$; then we have

$$f(x+hx, y+hy, \dots) = \log\left\{u_m(1+h)^m\right\} = \log u_m + m\log(1+h)$$

but, by Taylor's Theorem,

$$f(x+hx, y+hy, \dots) = \log u_m + h\left(x\frac{d}{dx} + y\frac{d}{dy} + \dots\right)\log u_m$$
$$+ \frac{h^2}{1.2}\left(x\frac{d}{dx} + y\frac{d}{dy} + \dots\right)^2\log u_m + \&\text{c.};$$

therefore $\log u_m + m\log(1+h) = \log u_m + h\left(x\frac{d}{dx} + \&\text{c.}\right)\log u_m + \&\text{c.}$;

therefore $$m\left\{h - \frac{h^2}{2} + \frac{h^3}{3} - \dots + (-)^{n+1}\frac{h^n}{n} + \dots\right\} =$$
$$h\left(x\frac{d}{dx} + \dots\right)\log u_m + \dots + \frac{h^n}{\lfloor n}\left(x\frac{d}{dx} + \dots\right)^n\log u_m + .$$

hence, equating coefficients of h^n, we obtain

$$\left(x\frac{d}{dx} + y\frac{d}{dy} + \dots\right)\log u_m = (-)^{n+1}m\,\lfloor n-1.$$

1493. (Proposed by Dr. SALMON, F.R.S.)—If a conic have double contact with two fixed conics, find the envelope of its chords of intersection with another fixed conic having double contact with one of the two fixed conics.

Solution by the PROPOSER; *and* F. D. THOMSON, M.A.

Let the two fixed conics be $S = 0$, $S + xy = 0$; then the variable conic

$$k^2x^2 + 2k(2S + xy) + y^2 = 0, \quad \text{or} \quad (kx + y)^2 + 4kS = 0,$$

and its intersections with $S + z^2 = 0$ are $kx + y \pm 2z\sqrt{(k)} = 0$, whose envelope is $xy = z^2$.

If, however, instead of touching the conic $S + xy$, the condition be that the conic should pass through two fixed points, let the variable conic be $S - (\lambda x + \mu y + \nu z)^2$; then we have the two conditions

$$\lambda x' + \mu y' + \nu z' = \sqrt{(S')}, \quad \lambda x'' + \mu y'' + \nu z'' = \sqrt{(S'')};$$

whence it follows that the line

$$\lambda x + \mu y + \nu z \pm (ax + by + cz)$$

passes through one of four fixed points.

The reciprocals of these questions are as follows: viz., If a conic touch doubly two fixed conics, the locus of the intersection of common tangents with another also having double contact with one of the fixed conics is a conic; except the second fixed conic breaks up into right lines, when the locus also breaks up into right lines.

1721. (Proposed by G. O. HANLON.)—Find the locus of a point such that, if perpendiculars are drawn therefrom on the sides of a quadrilateral, a circle will pass through the feet of these perpendiculars; and show geometrically how to find points on the locus; also determine the quadrilateral in order that the locus may be a conic.

I. *Solution by* T. A. HIRST, F.R.S.

The problem is obviously equivalent to finding the locus of the foci of a system of conics inscribed in the given quadrilateral. Since there is but one conic of such a system which touches an arbitrary line, the locus required (see my Solution of Question 1717, *Reprint*, Vol. IV. p. 19) must be a *circular* cubic passing through the six intersections of the sides of the complete quadrilateral. In order that this locus may break up into a *conic* and a right line, the latter constituent must be at infinity, and on it must be situated, necessarily, a pair of opposite intersections of the quadrilateral. In other words, the latter must be a parallelogram.

II. *Solution by* W. S. BURNSIDE, B.A.; *the* PROPOSER; J. DALE; *and others.*

As it is well known that the feet of the perpendiculars from the foci of a

conic on the tangents lie on a circle, the locus in question is that of the foci of conics touching four lines. Let the equations of the lines be $x = 0$, $y = 0$, $z = 0$, $\alpha x + \beta y + \gamma z = 0$; then the equation of the conic is of the form

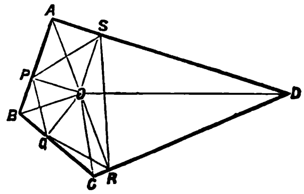
$$(lx)^{\frac{1}{2}} + (my)^{\frac{1}{2}} + (nz)^{\frac{1}{2}} = 0, \text{ where } l\alpha^{-1} + m\beta^{-1} + n\gamma^{-1} = 0;$$

also (see the second Solution of Quest. 1717, *Reprint*, Vol. IV. p. 20.)

$$l : m : n = (y^2 + z^2 + 2yz \cos A)x : (z^2 + x^2 + 2zx \cos B)y : (x^2 + y^2 + 2xy \cos C)z;$$

hence the equation of the required locus is

$$(y^2 + z^2)\frac{x}{\alpha} + (z^2 + x^2)\frac{y}{\beta} + (x^2 + y^2)\frac{z}{\gamma} + 2xyz\left(\frac{\cos A}{\alpha} + \frac{\cos B}{\beta} + \frac{\cos C}{\gamma}\right) = 0.$$



[Points in the locus may be easily constructed from the following property. Let ABCD be the quadrilateral, O a point on the required locus, and P, Q, R, S the feet of the perpendiculars from O on the sides of the quadrilateral. Then $\angle$ AOB + COD = ASP + BQP + CQR + DSR = 2 right-angles, since PQR + PSR = 2 right-angles. Hence if two circular segments be drawn on AB, CD, so as to contain supplemental angles, the arcs of these segments will intersect in points on the required locus. From the same property, moreover, the Cartesian equation of the locus may be readily obtained. In another form, the equation of the locus is given in Salmon's *Conics*, 4th ed., p. 261, ex. 15.]

1629. (Proposed by Dr. Booth, F.R.S.)—To investigate the geometrical signification of e, the base of the Napierian logarithms.

Solution by the Proposer.

Let $\Pi\,(m, \theta)$ denote the arc of a parabola, measured from the vertex to the point of the curve at which, a tangent being drawn, the focal perpendicular on it makes the angle θ with the axis. Let m be the focal distance of the vertex, or ***modulus***, (we shall see presently the propriety of this definition,) then by an elementary formula for rectification, we shall have

$$\Pi\,(m, \theta) = m \sec\theta \tan\theta + m\int \sec\theta d\theta \ldots\ldots\ldots\ldots\ldots (1).$$

Since $m \sec\theta \tan\theta$ is the projection of the radius vector on the tangent, it may be called for brevity the ***protangent***, and we have by common division,

$$\int \sec\theta d\theta = \int (1 - \sin^2\theta)^{-1} d(\sin\theta) = \sin\theta + \tfrac{1}{3}\sin^3\theta + \tfrac{1}{5}\sin^5\theta + \tfrac{1}{7}\sin^7\theta + \&c. (2).$$

In (1), if we make m equal to unity, and assume that the difference between the parabolic arc and its protangent is also equal to unity, we shall have $\Pi\,(1, \theta) - \sec\theta \tan\theta = 1$, and therefore

$$\int \sec\theta\, d\theta = 1 = \sin\theta + \tfrac{1}{3}\sin^3\theta + \tfrac{1}{5}\sin^5\theta + \&c. \ldots\ldots\ldots\ldots (3)$$

and reversing this series by known methods, we shall have

$\sin\theta = \dfrac{e^1 - e^{-1}}{e^1 + e^{-1}}$, e being the number called the base of the Napierian

logarithms; or, solving for e,

$$\sec\theta+\tan\theta=e \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).*$$

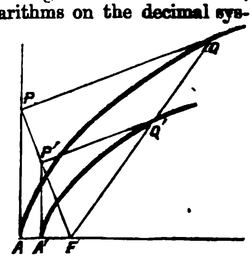
More generally, if we take the difference between the parabolic arc and its protangent equal to n, we shall have

$$\sec\theta_1+\tan\theta_1=e^n \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Hence, if on the modulus m of a parabola we describe a *Logocyclic curve*, or the curve whose polar equation is $r=m\,(\sec\theta\pm\tan\theta)$, and draw a tangent to the parabola perpendicular to this radius vector, the difference between the parabolic arc and its *protangent* will be the logarithm of the number $\sec\theta+\tan\theta$ and of its reciprocal $\sec\theta-\tan\theta$, when $m=1$.

Hence also, if we describe a series of confocal and coaxal parabolas, these parabolas may represent by their arcs as many systems of logarithms. Thus, if we take the parabola whose modulus $m=\cdot4342$, we shall have the parabola which represents the decimal system of logarithms.

Let D and Δ be the differences between the arcs of any two parabolas and their corresponding protangents to the same angle θ, the moduli being m and 1; then, as the confocal parabolas are similar figures and similarly placed, we shall have by similar triangles $D:\Delta=m:1$. Draw the radius vector $\sec\theta+\tan\theta=10$, and take the logarithms in the Napierian system; then $\Delta=\log 10$, and $D=1$, since D is the decimal logarithm of the base 10; hence $m\log 10=1$. Again, if we take the logarithms on the decimal system, draw $\sec\theta+\tan\theta=e$; then $\Delta=1$ and $D=\log e$, whence $m=\log e$. In the figure, since, FPQ and FP′Q′ are similar triangles, we have $AQ-QP:A'Q'-Q'P'=1:m$. Hence, also, as all these confocal parabolas are similar figures and are to each other as their parameters or moduli, it follows that all the logarithms, or tangential differences of the same number or logocyclic radius vector, are to each other as their moduli. And further, as the logocyclic curve admits of no negative radius vector, negative numbers can have no logarithms.



And if we change $\sec\theta$ into $\cos\theta$, and $\tan\theta$ into $i\sin\theta$, we shall find the logocyclic changed into a circle whose radius is $r=\cos\theta\pm i\sin\theta$, whence it follows that *imaginary* numbers have circular logarithms, while real numbers have parabolic or real logarithms.

1677. (Proposed by C. Taylor, M.A.)—Prove that a chord of constant length, drawn through either point of contact of a circle which has double contact with a fixed conic, envelops a confocal conic.

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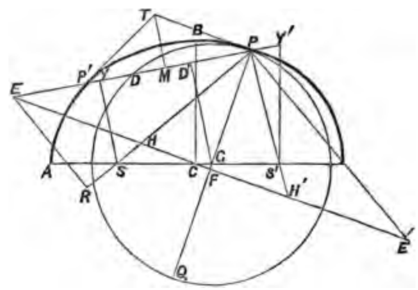
* Since $\quad \log\left(\frac{1+x}{1-x}\right)=2\,(x+\frac{1}{3}x^3+\frac{1}{5}x^5+\frac{1}{7}x^7+\&c.),$

we have $\quad \log\left(\frac{1+\sin\theta}{1-\sin\theta}\right)=2\,(\sin\theta+\frac{1}{3}\sin^3\theta+\frac{1}{5}\sin^5\theta+\&c.)=2,$

or $\quad \left(\frac{1+\sin\theta}{1-\sin\theta}\right)=\frac{(1+\sin\theta)^2}{\cos^2\theta}=e^2,$ or $\sec\theta+\tan\theta=e.$

Solution by J. DALE.



Let CA, CB be the semi-axes of the given conic, S, S′ its foci; PDQ a circle having double contact with this conic, P being one of the points of contact; and PD a chord of given length. From G, the foot of the normal PG, which is the centre of the circle, draw GD′ perpendicular to PD, then PD′ ($=\frac{1}{2}$PD) is given. Through the centre C draw the diameter conjugate to CP (that is to say, parallel to the tangent at P) meeting PD, PS, PS′ in E, H, H′. Then, by known properties, $PH = PH' = CA$; and from the similar triangles PGD′, PEF, we have $PG : PD' = PE : PF$, therefore $PE \,.\, PD' = PG \,.\, PF = CB^2$; hence PE is of constant magnitude.

Draw SY, S′Y′ perpendicular to PE, and ER perpendicular to ES; then, by similar triangles, we have

$PE : ER = PS : SY,\ ER : EH = PF : PH, \therefore PE : EH = PS.PF : SY.PH;$

Similarly $\qquad PE : EH' = PS' \,.\, PF : S'Y' \,.\, PH';$

therefore $\qquad PE^2 : EH \,.\, EH' = PS \,.\, PS' \,.\, PF^2 : SY \,.\, S'Y' \,.\, PH^2.$

But $\qquad EH \,.\, EH' = EF^2 - FH^2 = PE^2 - PH^2;$

also $\qquad PS \,.\, PS' \,.\, PF^2 = CA^2 \,.\, CB^2$, and $P\,H^2 = CA^2$;

therefore $PE^2 : PE^2 - CA^2 = CB^2 : SY \,.\, S'Y'$; hence the rectangle $SY \,.\, S'Y'$ is *constant*, and therefore PE is a tangent to a conic whose foci are S, S′. If CA′, CB′ are the semi-axes of this conic, we have

$$CB'^2 = S'Y \,.\, S'Y' = \frac{CB^2}{PE^2}(PE^2 - CA^2) = CB^2 - \left(\frac{PD \,.\, CA}{2CB}\right)^2,$$

$$CA'^2 = CA^2 - CB^2 + CB'^2 = \frac{CA^2}{PE^2}(PE^2 - CB^2) = CA^2 - \left(\frac{PD \,.\, CA}{2CB}\right)^2.$$

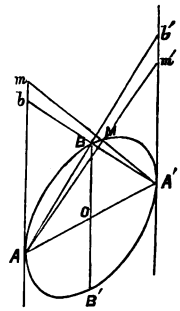
Let PE cut the fixed conic in P′, and at P′ draw a tangent meeting the tangent at P in T; then T lies on the normal to the enveloped conic, through the point of contact, and if TM be drawn perpendicular to PD, M is the point where PD touches the enveloped conic.

[The converse of this theorem, viz., "If PE be a tangent to a confocal conic, the line PE cut off by the parallel CE is of constant magnitude," forms Question 2026 of the *Lady's and Gentlemen's Diary* for 1864.]

1696. (Proposed by R. TUCKER, M.A.)—Supplemental chords of an ellipse are produced to meet the tangents at the extremities of the corresponding diameter; prove that the rectangle contained by the intercepts on the tangents is constant.

Solutions (1) *by* ARCHER STANLEY; (2) *by* ALPHA *and* J. DALE.

1. The connectors of A and A′ with the several points M of the conic constitute homographic pencils, and accordingly the points m, m', where these connectors intersect the parallel tangents at A and A′, constitute homographic ranges. To the point of contact of either of these tangents corresponds, manifestly, the point at infinity on the other. Hence, if BB′ be the diameter conjugate to AA′, and b, b' the points which correspond to B, the anharmonic ratios $[\mathrm{A}\,\infty m b]$, $[\infty\,\mathrm{A}'m'b']$ will be equal; whence we may at once deduce the relation



$$\mathrm{A}m\,.\,\mathrm{A}'m' = \mathrm{A}b\,.\,\mathrm{A}'b' = (\mathrm{BB}')^2 = \text{a constant.}$$

The above demonstration is essentially the same as that given by Chasles, at p. 119 of his *Traité des Sections Coniques*.

2. *Otherwise:* Referred to the conjugate semi-diameters OA′ ($= a$) and OB ($= b$) as axes, let the coordinates of M be (h, k); then the equations of AMm', A′Mm are

$$k\,(x+a) = (h+a)\,y \ldots . (\mathrm{A}m'); \quad k\,(x-a) = (h-a)\,y \ldots . (\mathrm{A}'m).$$

Making $x = a$ in (Am') and $= -a$ in (A′m), we have

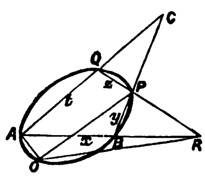
$$\mathrm{A}'m' = \frac{2ak}{a+h}, \quad \mathrm{A}m = \frac{2ak}{a-h}; \quad \therefore\ \mathrm{A}m\,.\,\mathrm{A}'m' = \frac{4a^2k^2}{a^2-h^2}.$$

But $b^2h^2 + a^2k^2 = a^2b^2$, or $a^2k^2 = b^2\,(a^2-h^2)$; $\therefore\ \mathrm{A}m\,.\,\mathrm{A}'m' = b^2$.

1586. (Proposed by X. U. J.)—If a straight line meet the sides BC, CA, AB of a triangle in P, Q, R respectively, the tangents at O to the conics OAPBQ, OAPCR form with the straight lines OA, OP a harmonic pencil. Also, if a fourth harmonic CK be drawn to the pencil CA, CO, CP, and a fourth harmonic RK to the pencil RA, RO, RP, these will intersect upon the tangent at O to the conic OAPBQ; and if a fourth harmonic be now drawn to KC, KO, KR, it will be *invariable* with respect to the three conics OAPBQ, OBQRC, OCRAP.

Solution by F. D. THOMSON, M.A.

Let $x = 0$, $y = 0$, $z = 0$, $t = 0$, be the equations to AB, BP, PQ, QA respectively, where (l, m, n, r being constants) x, y, z, t are connected by the equations



$$x+y+z+t = 0, \quad lx+my+nz+rt = 1.$$

Then the equation to a conic round ABPQ is of the form $xz = kyt$, and if this pass through the point O, whose coordinates are (x', y', z', t') suppose, we

must have $x'z' = ky't'$; hence the equation to the conic OABPQ is $\frac{xz}{x'z'} = \frac{yt}{y't'}$; and therefore the equation to the tangent at O is

$$\frac{x}{x'} + \frac{z}{z'} = \frac{y}{y'} + \frac{t}{t'}, \text{ or } \left(\frac{x}{x'} - \frac{t}{t'}\right) - \left(\frac{y}{y'} - \frac{z}{z'}\right) = 0 \ldots\ldots\ldots\ldots (\alpha).$$

Again, the equation to the conic OAPCR is $\frac{xy}{x'y'} = \frac{zt}{z't'}$; and therefore the equation to the tangent at O is

$$\frac{x}{x'} + \frac{y}{y'} = \frac{z}{z'} + \frac{t}{t'}, \text{ or } \left(\frac{x}{x'} - \frac{t}{t'}\right) + \left(\frac{y}{y'} - \frac{z}{z'}\right) = 0 \ldots\ldots\ldots\ldots (\beta).$$

Now $\frac{x}{x'} - \frac{t}{t'} = 0$ is the equation to OA, and $\frac{y}{y'} - \frac{z}{z'} = 0$ is the equation to OP; hence, by (α) and (β) we see that OA, OP, and the tangents at O form a harmonic pencil.

Again, the equation to CO is $\frac{y}{y'} - \frac{t}{t'} = 0$,

herefore the equation to CK is $\frac{y}{y'} + \frac{t}{t'} = 0$;

similarly the equation to RK is $\frac{x}{x'} + \frac{z}{z'} = 0$;

therefore, by (α), K is a point on the tangent at O to the conic OABPQ. The fourth harmonic to KC, KO, KR will be the line

$$\left(\frac{x}{x'} + \frac{z}{z'}\right) + \left(\frac{y}{y'} + \frac{t}{t'}\right) = 0, \text{ or } \frac{x}{x'} + \frac{y}{y'} + \frac{z}{z'} + \frac{t}{t'} = 0 \ldots\ldots (\gamma).$$

This equation (γ) involves x, y, z, t symmetrically, and would therefore have been arrived at if we had started with either of the conics OBQCR or OCRAP.

1707. (Proposed by T. COTTERILL, M.A.)—If A, B, C are the distances of a line from the angles of a triangle which it cuts in the points d, e, f respectively; then

$$\frac{A(B-C)}{ef} = \frac{B(C-A)}{fd} = \frac{C(A-B)}{de} = \frac{(B-C)(C-A)(A-B)}{2k},$$

k being the area of the triangle. The envelope of the lines in which $\lambda . ef + \mu . fd + \nu . fe = 0$ is a parabola; λ, μ, ν being as the coordinates of its axis. This equation can be written in such shapes, that an orthogonal substitution gives the envelope of the tangent at the vertex, and their form shows that they are also the envelope of the asymptotes of the rectangular hyperbolas circumscribing the triangle. The intercept of such a tangent between two sides of the triangle is equal to the projection of the remaining side upon it.

Solution by the PROPOSER.

Let a, b, c be the inclinations of the line (A, B, C) to the sides a, b, c. In any triangle, the product of a side and the sines of the adjacent angles is equal to the product of the perpendicular upon it from the opposite angle and the sine of that angle. Applying this to the triangles Aef, Bfd, Cde, we have $ef \sin b \sin c = \sin A \,.\, A$; $fd \sin c \sin a = \sin B \,.\, B$; $de \sin a \sin b = \sin C \,.\, C$. But $B - C = a \sin a$; $C - A = b \sin b$; $A - B = c \sin c$.

Hence eliminating the sines, and reducing, we have

$$\frac{ef}{A(B-C)} = \frac{fd}{B(C-A)} = \frac{de}{C(A-B)} = \frac{2k}{(B-C)(C-A)(A-B)}.$$

Among the many applications which may be made of this formula, let us consider the equation $\lambda \,.\, ef + \mu \,.\, fd + \nu \,.\, de = 0$, or for shortness $\Sigma\lambda \,.\, ef = 0$, where λ, μ, ν are constant, and therefore as the ordinates of a certain line $(A_m B_m C_m)$. Then the equation becomes $\Sigma A_m A (B-C) = 0$, or $\Sigma a \sin am \,.\, BC = 0$, from which, if $B + C - A = X$, &c., we obtain $\Sigma a \sin am \,.\, X^2 = 0$, the equations to a parabola referred, 1st, to a circumscribing; 2ndly, to a self-conjugate triangle. By taking the emanant, it will be seen that the point of contact with the line infinity is in the direction of m; i.e. m may be taken as the axis. Other forms arise from observing that $\Sigma A_m \,.\, ef = 0$ and $\Sigma ef = 0$ give the equations $\dfrac{ef}{a \sin am} = \dfrac{fd}{b \sin bm} = \dfrac{de}{c \sin cm}$. Combining these equations with the values of ef, fd, de, just given, we have

$$\frac{a \sin a \,.\, A}{a \sin am} = \frac{b \sin b \,.\, B}{b \sin bm} = \frac{c \sin c \,.\, C}{c \sin cm} = \frac{\Sigma A^2 (B-C)}{\Sigma a \sin am \,.\, A} =$$

$$-\frac{(B-C)(C-A)(A-B)}{ab \sin C \sin tm} = -2R \frac{\sin a \sin b \sin c}{\sin tm};$$

Collecting our results, we have $\Sigma a \sin am \,.\, BC = 0$; $\Sigma a \sin am \,.\, X^2 = 0$, and

$$\frac{\sin a \,.\, A}{\sin am} = \frac{\sin b \,.\, B}{\sin bm} = \frac{\sin c \,.\, C}{\sin cm} = -2R \frac{\sin a \sin b \sin c}{\sin tm},$$

eight equations to the same parabola. One look at them is sufficient to show that by the orthogonal substitution of the tangent at the vertex for the axis (or m), we obtain the equation to its envelope. For, in this case, $\sin am$, $\sin bm$, $\sin cm$, become $\cos a$, $\cos b$, $\cos c$, and $\sin tm = 1$. Hence $\Sigma a \cos a \,.\, BC = 0$; $\Sigma a \cos a \,.\, X^2 = 0$; and $\tan a \,.\, A = \tan b \,.\, B = \tan c \,.\, C = -2R \sin a \sin b \sin c$, must be the equations to the envelope of the tangents at the vertices of the parabolas inscribed in the triangle of reference. The curve is now well known to the readers of this Journal to be the tricusped hypocycloid, or tripod, as I shall venture to call it.

The last equations bring the tripod to a certain extent within the domain of Goniometry, so that it can be shown that whilst all these equations are cases of ease of one identity, another is intimately connected with the five remarkable points on a tangent.

In the rectangular tangent (A′B′C′), $\cot a \,.\, A' = \cot b \,.\, B' = \cot c \,.\, C'$. Hence $AA' = BB' = CC'$, and the curve is also the envelope of the rectangular asymptotes of conics circumscribing the triangle.

The equation $A \sin tm = -2R \sin am \sin b \sin c$ is curious. It contains, in fact, two sets of variables. Thus $2R \sin am = AS$ (S the focus) is an equation to the parabola. What is its meaning, when A is on the curve? How can the tangential distances of A from the curve be at once found? Why is the limit of $(\sin tm \,.\, A)$, t being the tangent at infinity, constant

whatever is the position of A? From this equation, and $ef \sin b \sin c = \sin A \,.\, A$, we get $ef \sin tm = a \sin am$, and for the vertical tangent this becomes $ef = a \cos a$.

Thus the tripod which first picks out from the parabolas the tangents of least intercept, joins to any two of them the tangent of greatest intercept between them to form a generating triangle.

I leave it for the reader to remark the constant relations connecting the squares of the intercepts of rectangular tangents between any two tangents, first to the parabola and then to the tripod.

1684. (Proposed by J. CASEY, B.A.)—Prove by inversion the leading properties of the Lemniscate of Bernoulli, including those of its osculating circle and Saladini's property that it is the curve down which a particle will descend through any arc in the same time as it will take to descend through the chord of the arc. Show also that it possesses, in a modified way, anharmonic properties similar to those of the conic sections.

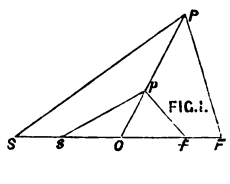
Solution by the PROPOSER.

1. Let Ox, Oy be the asymptotes of an equilateral hyperbola, PY a tangent at any point P, and OY a perpendicular on it from O; and let OY be produced to meet the hyperbola in Q; then it is easy to see that OP, OY are equally inclined to the axis, and that $OY \,.\, OP = OY \,.\, OQ = a^2$ (a being the semi-axis); therefore Q and Y are *inverse points* with respect to the circle whose radius is a and centre O. Now the central polar equation of the hyperbola being $\rho^2 \cos 2\theta = a^2$, that of the locus of Y is $a^2 \cos 2\theta = \rho^2$; hence this locus is the *Lemniscate of Bernoulli*; and corresponding to the asymptotes of the hyperbola are the inflectional tangents of the Lemniscate; in fact, they are the same lines.

2. Let s, f, (Fig. 1) be the inverse points of the foci S, F, and p the inverse of P; then by similar triangles, we have

$$SP : PO = sp : sO, \quad FP : PO = fp : fO;$$

but $SP \,.\, FP = PO^2$, hence $sp \,.\, fp = so \,.\, fo$; hence the lemniscate is the locus of the vertex of a triangle, whose base is given, and the rectangle under whose sides is equal to the square on half the base.



3. The rectangle contained by the intercepts made by a tangent on the asymptotes of the hyperbola is constant. Hence *if a circle pass through the origin and touch the lemniscate, the rectangle contained by the intercepts which it makes on the inflectional tangents is constant.*

4. The tangent and normal to a hyperbola bisect the angles between the focal radii vectores. Hence, (see Townsend's *Modern Geometry*, Vol. II., Chapter on Inversion.) *if through the origin and any point* P *in the lemniscate two circles be described passing through its two foci, of their two circles of inversion, one touches the curve at* P *and the other cuts it orthogonally at* P.

5. If any line intersect the hyperbola and its asymptotes, the intercepts between the hyperbola and asymptotes are equal. Hence, *if a circle passing through the origin* O *intersect the lemniscate in the points* b, c, *and the inflectional tangents in the points* a *and* d, *we have* $ab \cdot bd : ac \cdot cd = Ob^2 : Oc^2$.

6. The parallels to the tangent at the vertex of any diameter are bisected by that diameter. Hence, *if a series of circles touch each other at the centre of the lemniscate, the radii vectores drawn to it from the points where they cut the curve again form a pencil in involution, the double lines of which are the common tangent to the circles and the line drawn to the point of contact of that circle of the system which touches the lemniscate.*

7. If XPY be a tangent to the hyperbola, intersecting the asymptotes in X and Y, it is evident that $\angle XPO = 2POY$. Hence *if* OP' *be a radius vector to a lemniscate, and* $P'Y'$ *a tangent at* P' *intersecting the inflectional tangent in* Y', *then* $\angle OP'Y' = 2P'OY'$.

8. If B, C be two consecutive points in the hyperbola, and through B, C, perpendiculars be drawn to one of the asymptotes; then if OB be joined and produced to meet the parallel through C in D, it is easily seen that $BC = BD$. Hence if $OB'X$, $OC'X'$ (Fig. 2) be two semicircles through two consecutive points $B'C'$, and if OB', OC' be joined and OB' be produced to meet the semicircle $OC'X'$ in D', the element $B'C' = B'D'$. If therefore OX be vertical, and two material particles move under the action of gravity along OB', OC'; and if the particle which moves along OB', when it arrives at B', move with the same velocity along $B'C'$; the particles will both arrive at C' at the same instant. From this it readily follows that a material particle will arrive at C' in the same time whether it moves along the curve or chord. This is Saladini's beautiful property of the Lemniscate.

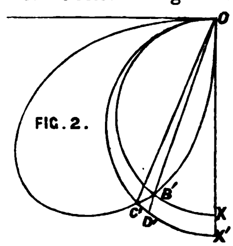


FIG. 2.

9. If PVQ be the circle of curvature at any point P, and if PV be the chord of curvature, then $PV = 2OP$. Hence *if* $P'V'Q'$ *be the osculating circle at any point* P' *of the Lemniscate, it divides the radius vector* OP' *in the ratio of* $1 : 2$.

10. Let OR (Fig. 3) be a radius vector to the hyperbola, RP' the tangent at R, OP' the perpendicular on RP', $P'X$ the tangent to the lemniscate at P', and OX the inflectional tangent. Now if $P'S$ be perpendicular to $P'X$, we have by Art. 7, $\angle OP'X = 2P'OX$, whence it follows that $\angle SP'O = SOP$; therefore OR is bisected in S, and the circle of curvature cuts $P'O$ in V' making $P'V' = \frac{2}{3}OP'$; hence it is seen at once that the centre of curvature at P' is the centre of gravity of the triangle $OP'R$, and that the radius of curvature is $\frac{1}{3}OR$.

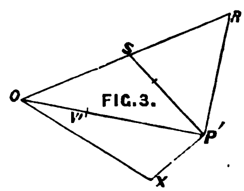


FIG. 3.

11. Hence the length of the evolute of the lemniscate between two points whose radii vectores are ρ_1, ρ_2 is $\frac{1}{3}a^2\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)$; for it $= \frac{1}{3}OR_1 - \frac{1}{3}OR_2$.

12. Since the perpendiculars at the extremities of the radii vectores of the lemniscate are tangents to the hyperbola, it follows that any two are cut homographically by all the rest.

13. If Q be a variable point, and A, B, C, D, four fixed points on the hyperbola, the pencil $\{Q . ABCD\}$ is constant. Hence *if Q′ be a variable and* A′B′C′D′ *four fixed points in the lemniscate, the four circles* OQ′A, OQ′B′, OQ′C′, OQ′D′, *intersect at a constant anharmonic ratio, and since the circles are coaxial, their centres are in a right line, and the anharmonic ratio of the four centres is given.*

It would be easy to multiply these properties, but enough has been given to show the great power of the method of inversion, and its utility in the Higher Geometry.

1766. (Proposed by J. Wilson.)—To find the condition necessary in order that one of two homofocal ellipses should be inscribed in a triangle, and the other circumscribed about it.

Solution by Professor Cremona.

Soient deux coniques rapportées au triangle conjugué commun

$$ax^2+by^2+cz^2=0, \qquad a'x^2+b'y^2+c'z^2=0.$$

Si la première est circonscrite à un triangle dans lequel le deuxième soit inscrite, tout point de la première est un sommet d'un triangle doué de la même propriété. Du point $(x=0,\ y:z=i\sqrt{c}:\sqrt{b})$ de la première conique menons à l'autre deux tangentes qui seront représentées par l'équation

$$(bc'-b'c)\,(a'x^2+b'y^2+c'z^2)-(ib'y\sqrt{c}+c'z\sqrt{b})^2=0,$$

et couperont la première conique en deux points situés sur la droite

$$i\sqrt{b}\,(ab'c'-a'bc'+a'b'c)y+\sqrt{c}\,(ab'c'+a'bc'-a'b'c)z=0.$$

Cette droite est tangente à la seconde conique si l'on satisfait à la condition

$$bc'(ab'c'-a'bc'+a'b'c)^2-b'c(ab'c'+a'bc'-a'b'c)^2=0,$$

c'est-à-dire $\quad a^2b'^2c'^2+b^2c'^2a'^2+c^2a'^2b'^2-2a'b'c'\,(bca'+cab'+abc')=0.$

C'est la condition pour que la première conique passe par les sommets d'un triangle circonscrit à la seconde.

Si les coniques proposées sont deux ellipses homofocales

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-e^2}-1=0, \qquad \frac{x^2}{a'^2}+\frac{y^2}{a'^2-e^2}-1=0,$$

il suffira de prendre

$$a:b:c=a^2-e^2:a^2:-a^2\,(a^2-e^2),$$
$$a':b':c'=a'^2-e^2:a'^2:-a'^2\,(a'^2-e^2);$$

et l'on aura pour la condition demandée

$$a^8-4a^6a'^2+6a^4a'^2e^2-4a^2a'^2e^4+a'^4e^4=0.$$

1737. (Proposed by Dr. BOOTH, F.R.S.)—Solve the equations
$$\alpha xy+\alpha' xz+a=0,\quad \beta yz+\beta' yx+b=0,\quad \gamma zx+\gamma' zy+c=0.$$

Solution by J. E. RADCLIFFE.

Putting $x^{-1}=\xi$, $y^{-1}=v$, $z^{-1}=\zeta$, $\xi v\zeta=\phi$, the equations become
$$\alpha\zeta+\alpha' v+a\phi=0\ldots(1),\ \ \beta\xi+\beta'\zeta+b\phi=0\ldots(2),\ \ \gamma v+\gamma'\xi+c\phi=0\ldots(3).$$
Then from P (1) + Q (2) + R (3) we have
$$(\mathrm{P}\alpha+\mathrm{Q}\beta')\,\zeta+(\mathrm{Q}\beta+\mathrm{R}\gamma')\,\xi+(\mathrm{P}\alpha'+\mathrm{R}\gamma)\,v+(\mathrm{P}a+\mathrm{Q}b+\mathrm{R}c)\,\phi=0.$$

Now to obtain the value say of ζ, let the coefficients of ξ and v be each put $=0$, or $\mathrm{Q}=-\mathrm{R}\gamma'\beta^{-1}$, $\mathrm{P}=-\mathrm{R}\gamma\alpha'^{-1}$; then, putting $\alpha\beta\gamma+\alpha'\beta'\gamma'=\Delta$, and $c\alpha'\beta-a\beta\gamma-b\alpha'\gamma'=\mathrm{C}$, &c., we have
$$\zeta=\frac{\mathrm{C}\phi}{\Delta},\ \text{in like manner}\ \xi=\frac{\mathrm{A}\phi}{\Delta},\ \text{and}\ v=\frac{\mathrm{B}\phi}{\Delta};$$
$$\therefore\ \xi v\zeta=\phi=\frac{\mathrm{ABC}\phi^3}{\Delta^3},\ \text{or}\ \phi=\frac{\Delta^{\frac{3}{2}}}{(\mathrm{ABC})^{\frac{1}{2}}};\ \therefore\ \xi=\frac{\mathrm{A}\phi}{\Delta}=\left(\frac{\mathrm{A}\Delta}{\mathrm{BC}}\right)^{\frac{1}{2}};$$
$$\text{hence}\quad x=\left(\frac{\mathrm{BC}}{\mathrm{A}\Delta}\right)^{\frac{1}{2}},\ y=\left(\frac{\mathrm{CA}}{\mathrm{B}\Delta}\right)^{\frac{1}{2}},\ z=\left(\frac{\mathrm{AB}}{\mathrm{C}\Delta}\right)^{\frac{1}{2}}.$$

Solution by ABRACADABRA; S. BILLS; J DALE; *and others.*

We may write the equations thus:—
$$\left.\begin{array}{rl}\alpha\,.\,xy+\alpha'\,.\,xz+ & \qquad\quad +a=0\\ \beta'\,.\,xy+ & \quad +\beta\,.\,yz+b=0\\ & \gamma\,.\,xz+\gamma'\,.\,yz+c=0\end{array}\right\};$$
$$\therefore\ \frac{xy}{\Sigma\pm(\alpha',\beta,c)}=\frac{xz}{\Sigma\pm(0,b,0)}=\frac{yz}{\Sigma\pm(a,\beta',\gamma')}=-\frac{1}{\alpha\beta\gamma+\alpha'\beta'\gamma'}$$
$$\therefore\ x=\left\{-\frac{[\Sigma\pm(\alpha',\beta,c)]\times[\Sigma\pm(0,b,0)]}{[\Sigma\pm(a,\beta',\gamma)]\times(\alpha\beta\gamma+\alpha'\beta'\gamma)}\right\}^{\frac{1}{2}}$$
$$y=\left\{-\frac{[\Sigma\pm(a,\beta',\gamma)]\times[\Sigma\pm(\alpha',\beta,c)]}{[\Sigma\pm(0,b,0)]\times(\alpha\beta\gamma+\alpha'\beta'\gamma')}\right\}^{\frac{1}{2}}$$
$$z=\left\{-\frac{[\Sigma\pm(0.\ b,0)]\times[\Sigma\pm(a,\beta',\gamma)]}{[\Sigma\pm(\alpha',\beta,c)]\times(\alpha\beta\gamma+\alpha'\beta'\gamma')}\right\}^{\frac{1}{2}}.$$

1712. (Proposed by the Rev. R. HARLEY, F.R.S.)—Let $f(x)$ be any rational integral function of x, of the degree n, and $f'(x)$ the first derived

function; then will $f'(x_1) \cdot f'(x_2) \dots f'(x_n) = a^{2-n} n^n \Delta$, where a is the coefficient of the highest power of x in $f(x)$, Δ is the discriminant, and $x_1, x_2, \dots x_n$ are the roots of $f(x) = 0$.

Solution by the PROPOSER.

Since, identically,

$$f(x) = a(x-x_1)(x-x_2) \dots\dots\dots\dots (x-x_n),$$

therefore, by differentiation, we have

$$\begin{aligned} f'(x) = & \, a(x-x_2)(x-x_3) \dots\dots\dots\dots (x-x_n) \\ & + a(x-x_1)(x-x_3) \dots\dots\dots\dots (x-x_n) \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & + a(x-x_1)(x-x_2) \dots\dots (x-x_{n-1}), \end{aligned}$$

$$f'(x_1) = a(x_1-x_2)(x_1-x_3) \dots\dots (x_1-x_n),$$

$$f'(x_2) = a(x_2-x_1)(x_2-x_3) \dots\dots\dots (x_2-x_n),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$f'(x_n) = a(x_n-x_1)(x_n-x_2) \dots (x_n-x_{n-1});$$

therefore $\qquad f'(x_1) \cdot f'(x_2) \dots f'(x_n) = (-)^{\frac{1}{2}n(n-1)} a^n \, \mathrm{S},$

where, for shortness, S is written in place of

$$(x_1-x_2)^2 (x_1-x_3)^2 \dots (x_2-x_3)^2 (x_2-x_4)^2 \dots$$

the product of the squares of the differences of the roots.

Now if Δ be expressed as a function of the coefficients of the equation $f(x) = 0$, and if l be the last or absolute term in this equation, we know that the expression for Δ will contain the term $+a^{n-1} l^{n-1}$; and since Δ is to a factor *près* equal to S, we may write $a^{2n-2} \, \mathrm{S} = k\Delta$, where k is a number. To determine this number, let us use the binomial equation

$$x^n - 1 = 0, \text{ for which } a^n \mathrm{S} = (-)^{n-1} k,$$

and therefore $\quad f'(x_1) \cdot f'(x_2) \dots f'(x_n) = (-)^{\frac{1}{2}n(n-1)+(n-1)} k.$

But in this case

$$f(x) = x^n - 1, \qquad f'(x) = nx^{n-1}, \qquad f'(x_1) = nx_1^{n-1}, \text{ \&c.};$$

therefore $\quad f'(x_1) \cdot f'(x_2) \dots f'(x_n) = n^n (x_1 x_2 \dots x_n)^{n-1};$

or, since $\qquad x_1 x_2 \dots x_n = (-1)^{n-1}$, we have

$$f'(x_1) \cdot f'(x_2) \dots f'(x_n) = (-)^{(n-1)^2} n^n = (-)^{n-1} n^n;$$

$$\therefore \; (-)^{\frac{1}{2}n(n-1)+(n-1)} k = (-)^{n-1} n^n; \qquad \text{whence } k = (-)^{\frac{1}{2}n(n-1)} n^n;$$

so that, in general,

$$a^{2n-2} \, \mathrm{S} = (-)^{\frac{1}{2}n(n-1)} n^n \Delta, \qquad \text{or } \mathrm{S} = (-)^{\frac{1}{2}n(n-1)} a^{-2n+2} n^n \Delta;$$

$$\therefore f'(x_1) \cdot f'(x_2) \dots f'(x_n) = (-)^{n(n-1)} a^{-n+2} n^n \Delta = a^{-n+2} n^n \Delta.$$

Since I proposed this question for solution, I have discovered that Prof. CAYLEY, in article 82 of his Fourth Memoir upon Quantics (*Phil. Trans.*

for 1858,) has determined the factor here denoted by k. I may mention, however, that the value of the discriminant given in that article requires to be slightly amended by writing a^{2m-2} in place of a^{m-2}, and that the same correction should be made in the last line on page 423.

The property established above is important on account of its connection with the theory of differential resolvents.

1724. (Proposed by W. K. CLIFFORD.)—The equations of three conics being given in the following forms: viz.,

$$a_1x^2+b_1y^2+c_1z^2+d_1w^2=0,\quad a_2x^2+\&c.=0,\quad a_3x^2+\&c.=0,$$

where $x+y+z+w\equiv 0$, show that a straight line $(\xi x+\eta y+\zeta z+\omega w=0)$ will be cut in involution by them, if

$$\Sigma\,.\begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \end{vmatrix}.(\xi-\eta)\,(\xi-\zeta)\,(\xi-\omega)\,.\,(\text{to four terms})=0.$$

Solution by W. S. BURNSIDE, B.A.

The condition that the lines given by the equations

$$Lx^2+2Mxy+Ny^2=0,\quad L_1x^2+2M_1xy+N_1y^2=0,\quad L_2x^2+2M_2xy+N_2y^2=0,$$

should form a pencil in involution is

$$\begin{vmatrix} L\,M\,N \\ L_1M_1N_1 \\ L_2M_2N_2 \end{vmatrix}=0,\ \text{or}\ (LM_1N_2)=0\ldots\ldots(\text{I.})$$

To apply this condition to the present Question, eliminate y and z from the equations of the conics by means of the equations $x+y+z+w=0$ and $x\xi+y\eta+z\zeta+w\omega=0$, and we shall find the equations of the lines joining (x,y) to the intersections of each conic with the line $x\xi+y\eta+z\zeta+w\omega=0$.

For shortness, let $\qquad \xi-\omega=\alpha,\quad \eta-\omega=\beta,\quad \zeta-\omega=\gamma\,;$

whence $\qquad x\xi+y\eta+z\zeta+w\omega=0$ becomes $\alpha x+\beta y+\gamma z=0\,;$

also let $\qquad (b_1c_2d_3)=A,\ (c_1d_2a_3)=B,\ (d_1a_2c_3)=C,\ (a_1b_2c_3)=D\,;$

then the condition (I.) is found to be

$$\begin{vmatrix} a_1\gamma^2+c_1\alpha^2+d_1(\gamma-\alpha)^2, & b_1\gamma^2+c_1\beta^2+d_1(\gamma-\beta)^2, & c_1\alpha\beta+d_1(\gamma-\alpha)(\gamma-\beta) \\ a_2\gamma^2+c_2\alpha^2+d_2(\gamma-\alpha)^2, & b_2\gamma^2+c_2\beta^2+d_2(\gamma-\beta)^2, & c_2\alpha\beta+d_2(\gamma-\alpha)(\gamma-\beta) \\ a_3\gamma^2+c_3\alpha^2+d_3(\gamma-\alpha)^2, & b_3\gamma^2+c_3\beta^2+d_3(\gamma-\beta)^2, & c_3\alpha\beta+d_3(\gamma-\alpha)(\gamma-\beta) \end{vmatrix}.$$

There are 18 determinants here, 12 of which vanish, and the remaining 6 are (when γ^2 is divided off, which was introduced before)

$$D\gamma^2\alpha\beta+C\gamma^2(\gamma-\alpha)(\gamma-\beta)+B\beta\alpha(\gamma-\beta)^2-B\beta^2(\gamma-\alpha)(\gamma-\beta)+$$
$$A\alpha\beta(\gamma-\alpha)^2-A\alpha^2(\gamma-\alpha)(\gamma-\beta)=0,$$

or $\ A\alpha(\alpha-\beta)(\alpha-\gamma)+B\beta(\beta-\gamma)(\beta-\alpha)+C\gamma(\gamma-\alpha)(\gamma-\beta)+D\alpha\beta\gamma=0,$

which is equivalent to the condition given in the Question.

1497. (Proposed by W. K. Clifford.)—(1.) Given three points by equations of the form $lx+my+nz=0$; prove that the area of the triangle contained by them is

$$(l_1m_2n_3) \div (l_1+m_1+n_1)(l_2+m_2+n_2)(l_3+m_3+n_3),$$

that of the triangle of reference being unity.

(2.) Also, if (123) denote the area of the triangle contained by the points 1, 2, 3, and so on, prove that

$$(123)(456) \equiv (156)(423)+(164)(523)+(145)(623).$$

Solution by the Proposer.

1. The area of the triangle formed by three points vanishes only when they are in a straight line, and becomes infinite only when one of them is at infinity. The condition that they may be in a straight line is

$$J \equiv \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0, \text{ or } (l_1m_2n_3)=0;$$

and the condition that one of them may be at infinity is

$$P \equiv (l_1+m_1+n_1)(l_2+m_2+n_2)(l_3+m_3+n_3)=0.$$

Now the expression for the area must be of no dimensions in the coefficients, since we are only concerned with their ratios; and the equation obtained by equating this expression to a constant must be of the first order in each set of coefficients; since, two of the points being fixed, the locus of the other is then a straight line. The expression for the area is therefore some numerical multiple of $\frac{J}{P}$. By putting $x,y,z=0$ for the three points, we find that the area of the fundamental triangle, on the same scale, is unity.

2. To prove the second proposition, take (456) for the fundamental triangle. Then, by applying the above interpretation to the well-known theorem

$$(l_1m_2n_3) \equiv l_1(m_2n_3)+m_1(n_2l_3)+n_1(l_2m_3)$$

we find it equivalent to

$$(123)(456) \equiv (156)(423)+(164)(523)+(145)(623),$$

the factor P dividing out on both sides.

1733. (Proposed by W. A. Whitworth, M.A.)—To find the area of a triangle the equations of whose sides in trilinear coordinates are

$$l_1\alpha+m_1\beta+n_1\gamma=0, \quad l_2\alpha+m_2\beta+n_2\gamma=0, \quad l_3\alpha+m_3\beta+n_3\gamma=0.$$

Solution by Abracadabra.

Let Δ' denote the required area, Δ the area of the triangle of reference, then (Salmon's Conics, 4th ed., Art. 39), we shall have

$$2\Delta' = \frac{\left\{\Sigma\pm(l_1\cos\alpha+\&\text{c.},\ l_2\sin\alpha+\&\text{c.}),\ -l_3p_\alpha-m_3p_\beta-\&\text{c.}\right\}^2}{\left\{\Sigma\pm(l_1\cos\alpha+\&\text{c.},\ l_2\sin\alpha+\&\text{c.})\right\}\left\{\Sigma\pm(l_2\cos\alpha+\&\text{c.},\ \&\text{c.})\right\}\left\{\Sigma\pm(\&\text{c.},\&\text{c.})\right\}}$$

$$\left(\text{or, putting } \mathrm{V} = \frac{\sin(\beta-\gamma)}{a} = \frac{\sin(\gamma-\alpha)}{b} = \frac{\sin(\alpha-\beta)}{c}\right)$$

$$= \frac{\left\{a\,.\,p_\alpha+b\,.\,p_\beta+c\,.\,p_\gamma\right\}^2 . \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}^2}{\mathrm{V}\left\{c\,.\begin{vmatrix} l_1, & m_1 \\ l_2, & m_2 \end{vmatrix} + a\,.\begin{vmatrix} m_1, & n_1 \\ m_2, & n_2 \end{vmatrix} + b\,.\begin{vmatrix} n_1, & l_1 \\ n_2, & l_2 \end{vmatrix}\right\}\left\{c\,.\begin{vmatrix} l_2, & m_2 \\ l_3, & m_3 \end{vmatrix} + \&\text{c.}\right\}\left\{c\,.\begin{vmatrix} l_3, & m_3 \\ l_1, & m_1 \end{vmatrix} + \&\text{c.}\right\}};$$

and hence twice the area of the triangle of reference is

$$2\Delta = \frac{\left\{a\,.\,p_\alpha+b\,.\,p_\beta+c\,.\,p_\gamma\right\}^2 l_1^2m_2^2n_3^2}{\mathrm{V}\,.\,abc\,.\,l_1^2m_2^2n_3^2}$$

$$\therefore\ \frac{\Delta'}{\Delta} = \frac{abc\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}^2}{\begin{vmatrix} a, & b, & c \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix} . \begin{vmatrix} a, & b, & c \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} . \begin{vmatrix} a, & b, & c \\ l_3, & m_3, & n_3 \\ l_1, & m_1, & n_1 \end{vmatrix}}.$$

II. *Solution by the* PROPOSER; M. JENKINS, B.A.; *and others.*

Let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be the vertices of the triangle Δ', expressed by trilinear coordinates; then in the Solution of Question 1497 (*Reprint*, Vol. I., p. 80), it is shown that

$$\Delta' = \frac{abc}{8\Delta^2}, \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \quad\cdots\cdots\cdots\cdots\cdots\cdots \quad \text{(I.)}$$

But x_1, y_1, z_1 are obtained by solving the simultaneous system

$$l_2x_1+m_2y_1+n_2z_1 = 0, \quad l_3x_1+m_3y_1+n_3z_1 = 0, \quad ax_1+by_1+cz_1 = 2\Delta;$$

and are therefore given by

$$\frac{x_1}{\begin{vmatrix} m_2 & n_2 \\ m_3 & n_3 \end{vmatrix}} = \frac{y_1}{\begin{vmatrix} n_2 & l_2 \\ n_3 & l_3 \end{vmatrix}} = \frac{z_1}{\begin{vmatrix} l_2 & m_2 \\ l_3 & m_3 \end{vmatrix}} = \frac{2\Delta}{\begin{vmatrix} a & b & c \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}}$$

and x_2, y_2, z_2; x_3, y_3, z_3 are given by similar equations; hence we have

$$\Delta' = \frac{abc\,\Delta \begin{vmatrix} \begin{vmatrix} m_2 & n_2 \\ m_3 & n_3 \end{vmatrix}, & \begin{vmatrix} n_2 & l_2 \\ n_3 & l_3 \end{vmatrix}, & \begin{vmatrix} l_2 & m_2 \\ l_3 & m_3 \end{vmatrix} \\ \begin{vmatrix} m_3 & n_3 \\ m_1 & n_1 \end{vmatrix}, & \begin{vmatrix} n_3 & l_3 \\ n_1 & l_1 \end{vmatrix}, & \begin{vmatrix} l_3 & m_3 \\ l_1 & m_1 \end{vmatrix} \\ \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}, & \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}, & \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a & b & c \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \cdot \begin{vmatrix} a & b & c \\ l_3 & m_3 & n_3 \\ l_1 & m_1 & n_1 \end{vmatrix} \cdot \begin{vmatrix} a & b & c \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}} \ldots\ldots \text{(II.)}$$

But the determinant forming the numerator of the last fraction is evidently equivalent to

$$\frac{1}{l_1} \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \cdot \begin{vmatrix} \begin{vmatrix} n_3 & l_3 \\ n_1 & l_1 \end{vmatrix}, & \begin{vmatrix} l_3 & m_3 \\ l_1 & m_1 \end{vmatrix} \\ \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}, & \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} \end{vmatrix}$$

And we find that

$$\therefore \begin{vmatrix} \begin{vmatrix} n_3 & l_3 \\ n_1 & l_1 \end{vmatrix} & \begin{vmatrix} l_3 & m_3 \\ l_1 & m_1 \end{vmatrix} \\ \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix} & \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} \end{vmatrix} \equiv l_1 \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}.$$

Hence the expression (II.) for the area (Δ') required becomes

$$\Delta' = abc\,\Delta \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}^2 \div \left\{ \begin{vmatrix} a & b & c \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} . \text{ \&c. \&c.} \right\} \ldots \text{(III).}$$

III. *Solution by* W. K. CLIFFORD.

Call the three lines 1, 2, 3. Then we have to find the area of the triangle included by the points (23), (31), (12); that is, by the points

$$L_1x + M_1y + N_1z = 0, \quad L_2x + M_2y + N_2z = 0, \quad L_3x + M_3y + N_3z = 0\,;$$

where $L_1 \ldots$ are the first minors of the determinant $(l_1m_2n_3)$. But by the Solution to Quest. 1497, this area is a numerical multiple of $\frac{J}{P}$, where

$$J \equiv (L_1M_2N_3) \equiv (l_1m_2n_3)^2$$

$$P \equiv (aL_1 + bM_1 + cN_1)\,(aL_2 + bM_2 + cN_2)\,(aL_3 + bM_3 + cN_3)$$

in the Trilinear system.

In the case of the fundamental triangle we find that $\frac{J}{P} = \frac{1}{abc}$. Hence the *ratio* has the value given in the foregoing Solutions.

The general formula for all systems (see my paper on "Analytical Metrics" in the *Quarterly Journal of Mathematics*, Vol. vii. p. 62) is

$$\frac{\{J(ABC)\}^2}{J(BC\infty).J(CA\infty).J(AB\infty)}$$

where J (ABC)=0 is the condition that the lines A, B, C, = 0 may meet in a point, and J (BC ∞)=0 is the condition that B and C may be parallel. The *ratio* to the area of the fundamental triangle may easily be found in any particular case by the method used above.

In the same way it may be shown that the volume of a tetrahedron is

$$\frac{\{J(ABCD)\}^3}{J(BCD\infty).J(CDA\infty).J(DAB\infty).J(ABC\infty)}$$

where J (ABCD)=0 is the condition that the planes A, B, C, D=0 may meet in a point, and J (BCD∞)=0 is the condition that B, C, D may be parallel to the same line.

In quadriplanar coordinates, for instance, if α, β, γ, δ, denote the areas of the faces of the fundamental tetrahedron, the equation to the plane at infinity is

$$\alpha x+\beta y+\gamma z+\delta w=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and the above expression for the volume, if calculated by means of (1), must be multiplied by the product $\alpha\beta\gamma\delta$ to give the ratio of the volume of the given tetrahedron to that of the fundamental one.

1732. (Proposed by T. A. HIRST, F.R.S.)—Prove that the characteristics (see Question 1573) of a system of conics, satisfying four conditions, remain unaltered when, in place of passing through a given point, each conic is required to divide a given finite segment harmonically.

Solution by W. K. CLIFFORD.

In a system of conics satisfying four conditions (Z, Z′, Z″, Z‴) let μ be the number of conics that pass through an arbitrary point, and ν the number that touch an arbitrary line. Suppose that the polars of a point P, in respect of all the conics of the system, envelope a curve of class x. Then from the point P, x tangents can be drawn to the curve; that is to say, there are x polars of P which pass through P. But a point which lies on its polar in respect of a given conic is a point on the conic. Therefore x conics pass through P. But μ conics (by hypothesis) pass through P; so that $x=\mu$. Thus we get Chasles' Prop. XII.,—*the polars of an arbitrary point, in respect of a system of conics* (μ, ν) *envelope a curve of class* μ. It follows that there are μ polars of P which pass through another arbitrary point Q; that is to say, there are μ conics of the system which divide harmonically a given segment PQ. This is Chasles' Prop. XXVIII.

Suppose now that the condition Z‴ is that the conics shall pass through a given point. Call S the condition that they shall divide harmonically the segment PQ. Then (by the above) the number of conics satisfying the conditions (Z, Z′, Z″, S), and passing through the given point, is μ; that is to say, the *first characteristic* (μ') of the system (Z, Z′, Z″, S) is equal to the *first characteristic* (μ) of the system (Z, Z′, Z″, Z‴) where Z‴ is the

condition of passing through a given point. In the next place let Z''' be the condition of touching a given line. Then the number of conics which satisfy the conditions (Z, Z', Z'', Z''', S) is the same as the number which satisfy the conditions (Z, Z', Z'', Z''', point); that is to say, the *second characteristic* (ν') of the system (Z, Z', Z'', S) is equal to the *second characteristic* (ν) of the system (Z, Z', Z'', point). Thus neither of the characteristics is altered when we substitute for the condition of passing through a given point, the condition S of dividing harmonically a given segment PQ.

By similar reasoning it may be shown that neither characteristic is altered when we substitute for the condition of touching a given line, the condition of subtending harmonically a given angle.

1709. (Proposed by H. McColl.)—1. Let $f(x)$ be any function of x, and let $f(a, b)$ express (according as a is greater or less than b) a superior or inferior limit to $f(x)$ for all values of x between a and b; show that if $f(a, b)$ has the same sign as $b-a$, no real root of the equation $f(x)=0$ exists between a and b.

2. Let $f(x) = A_n x^n + A_{n-1} x^{n-1} + \ldots + A_1 x + A_0$, n being a positive integer. Show that, when a and b are positive, $f(a, b)$ may be found by the following process. According as A_n is positive or negative, multiply it by a or b; add the product to A_{n-1}, and represent the result by k_1. According as k_1 is positive or negative, multiply it by a or b; add the product to A_{n-2} and represent the result by k_2. Proceeding in this way till $k_1, k_2, k_3, \ldots k_n$ are all found, we shall have $f(a, b) = k_n$.

Solution by the Proposer.

1. First, let $f(a, b)$ and $b-a$ be both positive. In this case, b being algebraically greater than a, the symbol $f(a, b)$ expresses an *inferior* limit to $f(x)$ for all values of x between a and b; so that, when x is thus restricted, $f(x)$ must always be greater than the positive quantity $f(a, b)$, and cannot, therefore, be equal to zero. Next, let $f(a, b)$ and $b-a$ be both negative. In this case, b being less than a, the symbol $f(a, b)$ expresses a *superior* limit to $f(x)$ for all values of x between a and b; so that when x is thus restricted, $f(x)$ must always be less (algebraically) than the negative quantity $f(a, b)$, and cannot, therefore, be equal to zero. The terms *superior* and *inferior* are, of course, used in their algebraic and not their numerical sense.

2. Let x_1 denote any value of x between a and b. Multiply A_n by x_1; add the product to A_{n-1}; and represent the result by r_1. Multiply r_1 by x_1; add the product to A_{n-2}; and represent the result by r_2. Proceeding thus by Horner's method, we shall ultimately have $r_n = f(x_1)$. First, let $f(a, b)$ express a superior limit to $f(x)$ for all values of x between a and b. In this case, a is greater than b; and it requires no formal demonstration to show that k_1 is greater (algebraically) than r_1; that k_2 is greater than r_2; and so on to k_n, which is greater than r_n or $f(x_1)$. Next, let $f(a, b)$ express an

inferior limit to $f(x)$ for all values of x between a and b. In this case, a is less than b; and it may be proved similarly that k_1 is algebraically less than r_1; that k_2 is less than r_2; and so on to k_n, which is less than r_n or $f(x_1)$.

N.B.—When $f(a)$ and $f(b)$ are positive, we must, in finding $f(a, b)$, make a represent the less quantity; but when $f(a)$ and $f(b)$ are negative, we must make a represent the greater quantity; otherwise, $f(a, b)$, when obtained, will not give the desired information.

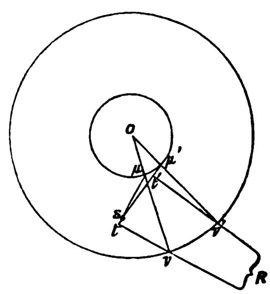
This method will frequently detect impossible roots by inspection. Take, for example, the equation $2x^4 - x^3 + 5x^2 - 3x + 4 = 0$. We find that $f(0, 1)$ is positive, which shows that no real root exists between 0 and 1; and next we find that $f(1, \infty)$ is positive, which shows that no real root exists between 1 and ∞.

$2-1+5-3+4 \;(0, 1$	$2-1+5-3+4 \;(1, \infty$
$\quad 0-1+0-3$	$\quad +2+1+6+3$
$-1+4-3+1$	$+1+6+3+7$

On the regular Tricusped Hypocycloid.

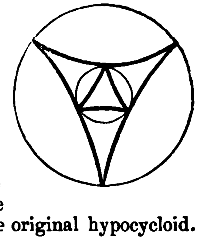
By M. Jenkins, B.A.

1. *Evolute.*—The evolute of the regular tricusped hypocycloid is another regular tricusped hypocycloid, and of *any* hypocycloid is a *similar* hypocycloid, lying between two circles (which may be called the pericentric and apocentric circles, since all hypocycloids are particular cases of planetary motion) such that the pericentric circle of the evolute coincides with the apocentric circle of the original hypocycloid, and the apse points of the evolute coincide with the cusp-points of the original hypocycloid.



It follows immediately from the ordinary definition of a hypocycloid, that the angle between two consecutive tangents has a constant ratio (κ, suppose) to the angle subtended by the intersections (μ, μ') of the tangents with the pericentric circle at the circumference of that circle; μ, μ' being the points of intersection with the circle nearest to s, the intersection of the tangents. This ratio in the case of the regular tricusp is one of equality (that is, $\kappa = 1$).

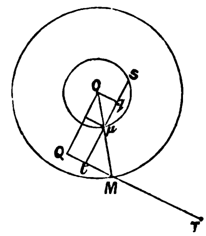
Let μt, $\mu' t'$ be two consecutive tangents cutting the pericentric circle in μ, μ', and each other in the point s. Draw $O\mu$, $O\mu'$ through the centre of the circle, meeting the apocentric circle in ν, ν'; and draw νt, $\nu' t'$ perpendicular to μt, $\mu' t'$ respectively. Then t, t' are the points of contact of the tangents μt, $\mu' t'$.



Let νt, $\nu' t'$ meet in R; then the locus of R is the evolute. Now νR, ν'R are consecutive tangents to the evolute, and νt, $\nu' t'$ are perpendicular to μt, $\mu' t'$ respectively; hence, for *any* hypocycloid

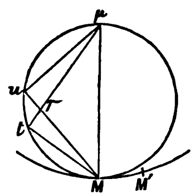
$$\angle \nu R \nu' = \mu s \mu' = \kappa \, . \, \tfrac{1}{2}\mu O \mu' = \kappa \, . \, \tfrac{1}{2}\nu O \nu'$$
$$= \kappa \, . \, \angle \text{ subtended at circumference by } \nu\nu';$$

therefore the locus of R is a *similar* hypocycloid having the apocentric circle of the original hypocycloid for its pericentric circle. And since a tangent to the evolute is a normal to the original hypocycloid, the apse points of the evolute are on the cusp-points of the original hypocycloid.

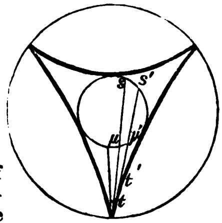


2. *Radius of Curvature*.—Let μt be a tangent, μ its centre, s its vertex, and t the point of contact; O the centre of the pericentric and apocentric circles; and M the point where Oμ cuts the apocentric circle. Then Mt is perpendicular to μt. Let OQ, parallel to μt, meet Mt in Q, make QT = 3QM, and draw Oq perpendicular to μs.

Then, since $\mu q = \frac{1}{2}\mu s = \frac{1}{2}\mu t$, $qt = 3\mu q$; and since tT is a tangent to the evolute, and QT = 3QM, therefore T is the point of contact of MT with the evolute; and QT = 3QM = 9Oq; hence tT (the radius of curvature of the original hypocycloid) = 8Oq.



3. *Length of Curve*.—Referring to the same figure as for radius of curvature, an element of the length of curve is equal to the product of the radius of curvature into the angle between the normals. Now if μM be a diameter of the rolling circle or hoop, μt a tangent to the hypocycloid, and we take an arc tu = MM′, where M′ is a contiguous point to M on the apocentric circle; we shall have $\angle t$Mu = 3 times $\angle$ between the normals to the hypocycloid through M, M′; and Mt = radius of curvature; hence if μt meet Mu in τ, element of curve $= \frac{4}{3}(\text{M}t \, . \, \angle t\text{M}\tau) = \frac{4}{3}t\tau = \frac{4}{3}$ difference of chords μt, μu; therefore the length of the curve between cusp and apse $= \frac{4}{3}$ diameter of rolling circle $= \frac{8}{3}r$, where r is the radius of the pericentric circle; and the whole length of the curve is 6 times that between cusp and apse = 16r.



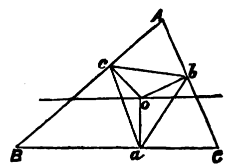
4. *Area*.—Let μt, $\mu' t'$ be two consecutive tangents; μ, μ' their centres; s, s' their vertices; and t, t' their points of contact; then we may suppose t, t' ultimately to coincide; sts', $\mu t \mu'$ to be triangles; $\mu s = \mu t$, and $\mu' s' = \mu' t'$; hence

$$\triangle \, \mu t \mu' = \tfrac{1}{4} sts' = \tfrac{1}{3} s\mu\mu' s'.$$

Proceeding from cusp to apse, the area between the pericentric circle and the hypocycloid, from the cuspidal tangent to the apse-point, is one-third of the semicircle; hence the area between the hypocycloid and the circle is equal to the area of the circle, and the whole area, including the circle, is double the area of the circle, or $2\pi r^2$.

1704. (Proposed by R. FULLERTON.)—From any point in a straight line perpendiculars are drawn on the sides of a triangle, and the feet of these perpendiculars joined; find the locus of the centre of gravity (or *centroid*) of the triangle thus formed.

Solution by J. DALE; E. FITZGERALD; *and others.*



Let O be any point in the straight line $l\alpha+m\beta+n\gamma=0$; then drawing Oa, Ob, Oc perpendicular to BC, CA, AB, the coordinates (x, y, z) of the centre of gravity of the triangle abc will be given by the following equations,

$$2\alpha+\cos C\,.\,\beta+\cos B\,.\,\gamma = 3x,$$
$$\cos C\,.\,\alpha+2\beta+\cos A\,.\,\gamma = 3y,$$
$$\cos \beta\,.\,\alpha+\cos A\,.\,\beta+2\gamma = 3z;$$

and we have given $l\alpha+m\beta+n\gamma=0$.

Eliminating α, β, γ, between these equations, we obtain

$$l\begin{vmatrix} x, & \cos C, & \cos B \\ y, & 2, & \cos A \\ z, & \cos A, & 2 \end{vmatrix} + m\begin{vmatrix} x, & \cos B, & 2 \\ y, & \cos A, & \cos C \\ z, & 2, & \cos B \end{vmatrix} + n\begin{vmatrix} x, & 2, & \cos C \\ y, & \cos C, & 2 \\ z, & \cos B, & \cos A \end{vmatrix} = 0.$$

This shows that the locus of the *centroid* is a straight line; its equation, in trilinear coordinates (x, y, z), may also be written as follows; viz.,

$$\left\{l(4-\cos^2 A)+m(\cos A\cos B-2\cos C)+n(\cos A\cos C-2\cos B)\right\}x$$
$$+\left\{l(\cos B\cos A-2\cos C)+m(4-\cos^2 B)+n(\cos B\cos C-2\cos A)\right\}y$$
$$+\left\{l(\cos C\cos A-2\cos B)+m(\cos C\cos B-2\cos A)+n(4-\cos^2 C)\right\}z=0.$$

[*Otherwise:* Taking the given line as axis of x, the coordinates of the feet of the perpendiculars drawn from a point $(h, 0)$ in it on the lines

$$\alpha_1 \equiv x\cos\alpha_1+y\sin\alpha_1-p_1=0,\quad \alpha_2=0,\quad \alpha_3=0,$$

that is, on the sides of the given triangle, are

$$x_1=h+(p_1-h\cos\alpha_1)\cos\alpha_1,\quad y_1=(p_1-h\cos\alpha_1)\sin\alpha_1;\ \ \&c.;\ \&c.;$$

hence the coordinates (x, y) of the centre of gravity of the triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) will be

$$x=\tfrac{1}{3}h\Sigma\sin^2\alpha+\tfrac{1}{3}\Sigma p\cos\alpha,\quad y=\tfrac{1}{3}\Sigma p\sin\alpha-\tfrac{1}{3}h\Sigma\sin\alpha\cos\alpha.$$

Eliminating h, we find that the locus of (x, y) is the straight line

$$(3x-\Sigma p\cos\alpha)\Sigma\sin\alpha\cos\alpha=(\Sigma p\sin\alpha-3y)\Sigma\sin^2\alpha.]$$

1715. (Proposed by J. GRIFFITHS, M.A.)—Let P be the point of intersection of the perpendiculars of any triangle ABC, O the centre of the circumscribed circle, M the middle point of PO, and O′ the centre of any one of the escribed circles of the triangles ABC, BPC, CPA, APB. Prove that the nine-point circle of the above triangle touches the common tangent of the circle (O′) and the circle described on MO′ as diameter.

Solution by J. DALE; H. MURPHY; *and others.*

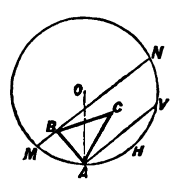
Here M is the centre of the nine-point circle of the triangle ABC, and it is known (see the solution of Mr. Wilkinson's Prize Question, in the *Lady's Diary* for 1855) that this circle touches each of the escribed circles above mentioned. Moreover the circle passing through the centres M, O′ of two circles which touch each other, and having its centre on the line MO′, touches the common tangents of these circles.

1333. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.)—Three points being taken at random within a sphere as the corners of a plane triangle, determine the probability that it shall be acute.

Solution by M. W. CROFTON, B.A.

We shall first prove that one of the points may be fixed on the surface of the sphere without altering the probability in question.

In any position of the three points, one of them is farthest from the centre; call its distance from the centre ρ; then the other two points are confined within a concentric sphere of radius ρ. Let P be the probability of an acute triangle when one point is thus confined to the surface of a sphere, radius ρ, and the other two within that sphere. This probability P is clearly independent of the value of ρ. Now, taking the given sphere, we shall get all the possible cases by making the *farthest point* move over every point of its volume, giving, in each of its positions, every possible position to the two nearer points. And *in each of its positions* the probability of the two nearer ones forming with it an acute triangle is P. Therefore the total probability is equal to P.



Let one of the points A be fixed on the surface of the sphere; if two other points B, C are taken at random inside it, we shall find separately the probability of each angle A, B, C being obtuse; as the events are mutually exclusive, the probability of the triangle ABC being *obtuse* will be the sum of these three probabilities.

1. To find the chance that A is obtuse, let us fix B; then, drawing the plane AV perpendicular to AB, the chance required is $\dfrac{\text{volume of segment AHV}}{\text{volume of sphere}}$.

Let $r = \text{OA} = $ radius of sphere, $\rho = \text{AB}$, $\theta = \angle\text{OAB}$; then segment AHV $= \frac{1}{3}\pi r^3(1-\cos\theta)^2(2+\cos\theta)$, therefore chance $= \frac{1}{4}(1-\cos\theta)^2(2+\cos\theta)$. Now, supposing B to move over the whole volume of the sphere, we have for the probability (P_A) that A is obtuse

$$\text{P}_\text{A} = \frac{1}{\text{sphere}}\iiint \tfrac{1}{4}(1-\cos\theta)^2(2+\cos\theta)\,d\text{V}$$

$$= \frac{3}{8r^3}\int_0^{\frac{1}{2}\pi}\int_0^{2r\cos\theta}(2-3\cos\theta+\cos^3\theta)\,\rho^2\sin\theta\,d\theta\,d\rho.$$

The integration is easily performed, and we find $\text{P}_\text{A} = \dfrac{3}{70}$.

2. To find the chance (P_B) that B is obtuse. Fix B as before; then the chance that B is acute is $\dfrac{\text{segment MHN}}{\text{sphere}}$.

Now, volume of MHN $= \frac{1}{3}\pi r^3\left(\dfrac{\rho}{r}+1-\cos\theta\right)^2\left(2+\cos\theta-\dfrac{\rho}{r}\right)$,

$$\therefore\ \text{chance} = \tfrac{1}{4}\left\{2-3\cos\theta+\cos^3\theta+3\frac{\rho}{r}(1-\cos^2\theta)+3\frac{\rho^2}{r^2}\cos\theta-\frac{\rho^3}{r^3}\right\}.$$

Hence the total probability $(1-P_B)$ that B is acute is, as before,

$$\frac{3}{8r^3}\int_0^{\frac{1}{2}\pi}\int_0^{2r\cos\theta}\left\{2-3\cos\theta+\cos^3\theta+3\frac{\rho}{r}(1-\cos^2\theta)+3\frac{\rho^2}{r^2}\cos\theta-\frac{\rho^3}{r^3}\right\}\rho^2\sin\theta\,d\theta\,d\rho,$$

whence, by very easy integrations, we find $1-P_B = \dfrac{53}{70}$, or $P_B = \dfrac{17}{70}$.

The probability that C is obtuse is of course the same as for B; hence the probability that the triangle is obtuse is

$$P_A+P_B+P_C = \frac{3}{70}+\frac{17}{70}+\frac{17}{70}=\frac{37}{70};$$

and therefore the probability that it is acute is $\dfrac{33}{70}$.

[Mr. Crofton's method may be readily applied to show that if three points are taken at random within a *circle*, the probability of their being the vertices of an acute triangle is $\dfrac{4}{\pi^2}-\dfrac{1}{8}$. Mr. Woolhouse's solution for the *sphere* is given on p. 234 of the *Educational Times* for January, 1863, and for the *circle*, on p. 22, Vol. I. of the *Reprint*.]

1692. (Proposed by H. J. Purkiss, B.A.)—Prove that, n being any positive integer,

$$\frac{2^{2n}}{\underline{|2n+1}} = \frac{1}{\underline{|2n+1}}+\frac{1}{\underline{|2n-1}}\frac{1}{\underline{|2}}+\frac{1}{\underline{|2n-3}}\frac{1}{\underline{|4}}+\cdots\cdots\cdots+\frac{1}{\underline{|2n}},$$

$$\frac{2^{4n-2}}{\underline{|4n}} = \frac{1}{2}\frac{1}{(\underline{|2n})^2}+\frac{1}{\underline{|2n-2}}\frac{1}{\underline{|2n+2}}+\frac{1}{\underline{|2n-4}}\frac{1}{\underline{|2n+4}}+\cdots+\frac{1}{\underline{|4n}},$$

$$= \frac{1}{\underline{|2n-1}}\frac{1}{\underline{|2n+1}}+\frac{1}{\underline{|2n-3}}\frac{1}{\underline{|2n+3}}+\cdots\cdots\cdots+\frac{1}{\underline{|4n-1}},$$

$$\frac{2^{4n}}{\underline{|4n+2}} = \frac{1}{2}\frac{1}{(\underline{|2n+1})^2}+\frac{1}{\underline{|2n-1}}\frac{1}{\underline{|2n+3}}+\cdots\cdots\cdots+\frac{1}{\underline{|4n+1}},$$

$$= \frac{1}{\underline{|2n}}\frac{1}{\underline{|2n+2}}+\frac{1}{\underline{|2n-2}}\frac{1}{\underline{|2n+4}}+\frac{1}{\underline{|2n-4}}\frac{1}{\underline{|2n+6}}+\cdots+\frac{1}{\underline{|4n+2}}.$$

Solution by the PROPOSER.

Substituting the series for $\sin 2\theta$, $\sin\theta$, and $\cos\theta$ in the formula $\sin 2\theta = 2\sin\theta\cos\theta$, and equating coefficients of θ^{2n+1}, we get the first result. In a similar way, starting with $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$, and equating coefficients of θ^{4n} and θ^{4n+2}, we get the four other results.

1750. (Proposed by W. K. CLIFFORD.)—Given four straight lines wh se equations are connected by the syzygy $x+y+z+w=0$; show that the straight lines

$$lx+my+nz+sw=0,\quad \lambda x+\mu y+\nu z+\sigma w=0,$$

will be conjugates in respect of any conic touching (x, y, z, w), if

$$(l\mu+\lambda m)+(n\sigma+\nu s) = (l\nu+\lambda n)+(s\mu+\sigma m) = (l\sigma+\lambda s)+(m\nu+\mu n).$$

Show also that if a quadrangle be formed from the quadrilateral by taking the pole of each line in respect of the triangle formed by the other three, then the relation between the two figures will be reciprocal; and if two straight lines be conjugates in respect of any conic inscribed in the quadrilateral, their poles in respect of the common connector-triangle will be conjugates in respect of any conic circumscribing the quadrangle.

Solution by W. S. BURNSIDE, B.A.

1. Let $L=0$, $M=0$, $N=0$ be the equations of the diagonals of the quadrilateral formed by the lines x, y, z, w; then we have $x = L-M-N$, $y = -L+M-N$, $z = -L-M+N$, giving $w = L+M+N$; also the equation of any conic inscribed in the quadrilateral is of the form $aL^2+bM^2+cN^2=0$, where $A+B+C=0$ (A, B, C, &c., being the coefficients of the reciprocal conic as usual; see Salmon's *Conics*, Arts. 278 and 287). Now, reducing $lx+my+nz+sw$ and $\lambda x+\mu y+\nu z+\sigma w$ to the forms $L\alpha+M\beta+N\gamma$ and $L\alpha'+M\beta'+N\gamma'$, we have $\alpha = l-m-n+s$, $\beta = -l+m-n+s$, $\gamma = -l-m+n+s$; and similarly, α', β', γ' are expressed in terms of $\lambda, \mu, \nu, \sigma$. The condition that these lines should be conjugates with respect to $aL^2+bM^2+cN^2=0$ is $A\alpha\alpha'+B\beta\beta'+C\gamma\gamma'=0$; hence by making $\alpha\alpha'=\beta\beta'=\gamma\gamma'$, we satisfy the last condition when $A+B+C=0$: but this is the relation given in the Question, as is seen by writing for α, β, γ and α', β', γ' their values; in fact, $\alpha\alpha'+\beta\beta' = (s-n)(\sigma-\nu)+(l-m)(\lambda-\mu) = (l\lambda+m\mu+n\nu+s\sigma)-(l\mu+m\lambda)-(n\sigma+\nu s)$, $\beta\beta'+\gamma\gamma' =$ &c., &c.

2. The coordinates (referred to L, M, N) of the four points forming the quadrangle derived from the quadrilateral, as indicated in the Question, are $(1,-1,-1)$, $(-1,1,-1)$, $(-1,-1,1)$, $(1,1,1)$; and the equations of the lines joining these points two and two are $L\pm M=0$, $M\pm N=0$, $N\pm L=0$. Now, considering the triangle formed from the first three points, the equations of its sides are $M+N=0$, $N+L=0$, $L+M=0$; and plainly the *polar* of the fourth point $(1, 1, 1)$ is given by the equation $L+M+N=0$ or $w=0$; and similarly for the other triangles; which proves the reciprocity of the quadrilateral and quadrangle.

3. The polars of the points $(1, \pm 1, \pm 1)$, with regard to the conic $L^2 + M^2 + N^2 = 0$, are the four lines $L \pm M \pm N = 0$; hence we have another reciprocal relation between the quadrangle and the quadrilateral. Moreover the pole of any line with regard to the conic $L^2 + M^2 + N^2 = 0$, and the *common connector* triangle (L, M, N) is the same; this being so, the third statement in the Question may be written as follows:—If two lines be conjugates with respect to any conic inscribed in the *quadrilateral*, their poles (in the ordinary sense) will be conjugates with respect to any conic circumscribing the *quadrangle*; the truth of which is manifest.

4. By means of the relations $(l\mu + \lambda m) + (n\sigma + \nu s) = (l\nu + \lambda n) + (s\mu + \sigma m)$ $= (l\sigma + \lambda s) + (m\nu + \mu n)$, combined with the identical relation between $\lambda, \mu, \nu, \sigma$ of the form $a\lambda + b\mu + c\nu + d\sigma = 0$, we may express $\lambda, \mu, \nu, \sigma$ in terms of l, m, n, s, involving the latter in the second degree; and hence it might be shown that when one of the lines passes through a fixed point, the *conjugate* line touches a conic inscribed in the triangle formed by the diagonals of the quadrilateral x, y, z, w. This also appears from writing the equations of the two lines in the forms $L\alpha + M\beta + N\gamma = 0$ and $\frac{L}{\alpha} + \frac{M}{\beta} + \frac{N}{\gamma} = 0$.

1752. (Proposed by Professor SYLVESTER.)—Prove that any proper fraction may be expanded under the form $\frac{e_1}{2} + \frac{e_2}{2^2} + \frac{e_3}{2^3} + \ldots + \frac{e_4}{2^4} + \ldots$. where the e's are all either zero or unity, and moreover become periodic after a certain point in the series; and show thereby that a calculating machine capable of multiplying and extracting square roots, will serve for the extraction of any root whatever.

Solution by E. L. ROSOLIS.

Let $\frac{a}{b}$ be any proper fraction, and let e_1 denote the integral part of $\frac{2a}{b}$, and $\frac{a_1}{b}$ the remainder; then $\frac{a}{b} = \frac{1}{2} \times \frac{2a}{b} = \frac{1}{2}\left(e_1 + \frac{a}{b}\right)$.

Again let e_2 be the integral part of $\frac{2a_1}{b}$, and $\frac{a_2}{b}$ the remainder; then

$$\frac{a_1}{b} = \frac{1}{2}\left(e_2 + \frac{a_2}{b}\right); \text{ and, in like manner, } \frac{a_2}{b} = \frac{1}{2}\left(e_3 + \frac{a_3}{b}\right), \text{ \&c.}$$

therefore $$\frac{a}{b} = \frac{e_1}{2} + \frac{{}_2}{2^2} + \frac{e_3}{2^3} + \ldots + \frac{e_n}{e^n} + \ldots,$$

where the e's are either zero or unity, since a is less than b, and they become periodic, when b is not a power of 2, just as ordinary fractions become circulating decimals in certain cases.

Hence $m^{\frac{1}{n}} = m^{\frac{e_1}{2}+\frac{e_2}{2^2}+\frac{e_3}{2^3}+\ldots} = \sqrt{m^{e_1}\sqrt{m^{e_2}\sqrt{m^{e_3}\ldots\ldots}}}$

therefore the nth root of a number m can be extracted by multiplying and extracting square roots.

1760. (Proposed by J. GRIFFITHS, M.A.)—Let each angular point of any triangle ABC be joined with each of two given points; and let α, β, γ; α', β', γ' denote the six points of intersection of the joining lines with the opposite sides of the triangle. It is required to prove that the conic which touches the sides of the two triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ touches also the two axes of homology of the triangles ABC, $\alpha\beta\gamma$, and ABC, $\alpha'\beta'\gamma'$. Show also that the triangle ABC is self-conjugate with respect to this conic.

What is the reciprocal of this theorem?

Solution by W. S. BURNSIDE, B.A.

Taking ABC as triangle of reference, the equations of the sides of the triangle $\alpha\beta\gamma$ and its axis of homology are of the form $lx \pm my \pm nz = 0$, and the conic $ax^2+by^2+cz^2=0$ will touch these four lines if $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=0$; similarly the same conic will touch the sides and axis of homology of the triangle $\alpha'\beta'\gamma'$ if $\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}=0$, the equations of these lines being $l_1x \pm m_1y \pm n_1z=0$; hence the conic given by the equation $\frac{x^2}{m^2n_1^2-m_1^2n^2}+\frac{y^2}{n^2l_1^2-n_1^2l^2}+\frac{z^2}{l^2m_1^2-l_1^2m^2}=0$ touches the eight lines $lx \pm my \pm nz=0$, $l_1x \pm m_1y \pm n_1z=0$; and with regard to this conic the triangle ABC is self-conjugate.

The reciprocal of this theorem is as follows:—

If, of the five quadrilaterals formed by five lines, two be taken; representing their common triangle by (abc), and the triangles formed by their diagonals respectively by $(a_1b_1c_1)$, $(a_2b_2c_2)$; a conic circumscribes the triangles $(a_1b_1c_1)$, $(a_2b_2c_2)$, and also passes through the two points, where the lines aa_1, bb_1, cc_1, and aa_2, bb_2, cc_2 are concurrent; the triangle (abc) being self-conjugate with respect to the conic.

1775. (Proposed by W. K. CLIFFORD.)—If a straight line meet the faces of the tetrahedron ABCD in the points a, b, c, d, respectively; the spheres whose diameters are Aa, Bb, Cc, Dd have a common radical axis.

Solution by PROFESSOR CAYLEY; *and* H. M. TAYLOR, B.A.

1. Let the given line be taken for the axis of z; the axes of x, y being any rectangular axes in the plane perpendicular thereto; the equations of the given line are therefore $(x=0, y=0)$. Take $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$, $(\alpha_4, \beta_4, \gamma_4)$ for the coordinates of the points A, B, C, D respectively; and $(0, 0, c_1)$, $(0, 0, c_2)$, $(0, 0, c_3)$, $(0, 0, c_4)$ for the coordinates of the points a, b, c, d respectively. Then, to determine c_1, the equation of the plane BCD is

$$\begin{vmatrix} x, & y, & z, & 1 \\ \alpha_2, & \beta_2, & \gamma_2, & 1 \\ \alpha_3, & \beta_3, & \gamma_3, & 1 \\ \alpha_4, & \beta_4, & \gamma_4, & 1 \end{vmatrix} = 0,$$

and cutting this by the line $x=0, y=0$, we have

$$\begin{vmatrix} 0, & 0, & c_1, & 1 \\ \alpha_2, & \beta_2, & \gamma_2, & 1 \\ \alpha_3, & \beta_3, & \gamma_3, & 1 \\ \alpha_4, & \beta_4, & \gamma_4, & 1 \end{vmatrix} = 0,$$

with similar equations for c_2, c_3, c_4 respectively. The four equations may be united into the single equation

$$\begin{vmatrix} c_1p_1, & 1, & \alpha_1, & \beta_1 \\ c_2p_2, & 1, & \alpha_2, & \beta_2 \\ c_3p_3, & 1, & \alpha_3, & \beta_3 \\ c_4p_4, & 1, & \alpha_4, & \beta_4 \end{vmatrix} = \begin{vmatrix} p_1, & \alpha_1, & \beta_1, & \gamma_1 \\ p_2, & \alpha_2, & \beta_2, & \gamma_2 \\ p_3, & \alpha_3, & \beta_3, & \gamma_3 \\ p_4, & \alpha_4, & \beta_4, & \gamma_4 \end{vmatrix},$$

where p_1, p_2, p_3, p_4 are arbitrary multipliers. Hence, writing successively $(p_1, p_2, p_3, p_4) = (1, 1, 1, 1)$ and $(p_1, p_2, p_3, p_4) = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$; we have *first*

$$\begin{vmatrix} c_1, & 1, & \alpha_1, & \beta_1 \\ c_2, & 1, & \alpha_2, & \beta_2 \\ c_3, & 1, & \alpha_3, & \beta_3 \\ c_4, & 1, & \alpha_4, & \beta_4 \end{vmatrix} = \begin{vmatrix} 1, & \alpha_1, & \beta_1, & \gamma_1 \\ 1, & \alpha_2, & \beta_2, & \gamma_2 \\ 1, & \alpha_3, & \beta_3, & \gamma_3 \\ 1, & \alpha_4, & \beta_4, & \gamma_4 \end{vmatrix},$$

that is,

$$\begin{vmatrix} 1, & \alpha_1, & \beta_1, & c_1+\gamma_1 \\ 1, & \alpha_2, & \beta_2, & c_2+\gamma_2 \\ 1, & \alpha_3, & \beta_3, & c_3+\gamma_3 \\ 1, & \alpha_4, & \beta_4, & c_4+\gamma_4 \end{vmatrix} = 0;$$

and *secondly*,

$$\begin{vmatrix} c_1\gamma_1, & 1, & \alpha_1, & \beta_1 \\ c_2\gamma_2, & 1, & \alpha_2, & \beta_2 \\ c_3\gamma_3, & 1, & \alpha_3, & \beta_3 \\ c_4\gamma_4, & 1, & \alpha_4, & \beta_4 \end{vmatrix} = \begin{vmatrix} \gamma_1, & \alpha_1, & \beta_1, & \gamma_1 \\ \gamma_2, & \alpha_2, & \beta_2, & \gamma_2 \\ \gamma_3, & \alpha_3, & \beta_3, & \gamma_3 \\ \gamma_4, & \alpha_4, & \beta_4, & \gamma_4 \end{vmatrix},$$

that is,

$$\begin{vmatrix} 1, & \alpha_1, & \beta_1, & c_1\gamma_1 \\ 1, & \alpha_2, & \beta_2, & c_2\gamma_2 \\ 1, & \alpha_3, & \beta_3, & c_3\gamma_3 \\ 1, & \alpha_4, & \beta_4, & c_4\gamma_4 \end{vmatrix} = 0;$$

and these two results may be united into the single formula

$$\left\|\begin{array}{lllll} 1, & \alpha_1, & \beta_1, & c_1+\gamma_1, & c_1\gamma_1 \\ 1, & \alpha_2, & \beta_2, & c_2+\gamma_2, & c_2\gamma_2 \\ 1, & \alpha_3, & \beta_3, & c_3+\gamma_3, & c_3\gamma_3 \\ 1, & \alpha_4, & \beta_4, & c_4+\gamma_4, & c_4\gamma_4 \end{array}\right\| = 0.$$

Now the equation of a sphere having for the extremities of a diameter the points (α, β, γ) and (a, b, c) is

$$[x-\tfrac{1}{2}(a+\alpha)]^2+[y-\tfrac{1}{2}(b+\beta)]^2+[z-\tfrac{1}{2}(c+\gamma)]^2 = \tfrac{1}{4}[(a-\alpha)^2+(b-\beta)^2+(c-\gamma)^2],$$

or, $$(x-a)(x-\alpha)+(y-b)(y-\beta)+(z-c)(z-\gamma) = 0,$$

or, $$x^2+y^2+z^2-(a+\alpha)x-(b+\beta)y-(c+\gamma)z+a\alpha+b\beta+c\gamma = 0;$$

therefore, when the two points are (α, β, γ) and $(0, 0, c)$, the equation is

$$x^2+y^2+z^2-\alpha x-\beta y-(c+\gamma)z+c\gamma = 0.$$

Hence, putting for shortness $P = -\alpha x-\beta y-(c+\gamma)z + c\gamma$, viz., $P_1 = -\alpha_1 x-\beta_1 y-(c_1+\gamma_1)z+c_1\gamma_1$, &c., the equations of the four spheres are

$$x^2+y^2+z^2+P_1 = 0, \quad x^2+y^2+z^2+P_2 = 0, \quad x^2+y^2+z^2+P_3 = 0,$$
$$x^2+y^2+z^2+P_4 = 0,$$

and the four spheres will have a common radical axis, if for proper values of the multipliers μ, ν, ρ we have

$$\mu(P_1-P_2)+\nu(P_1-P_3)+\rho(P_1-P_4) = 0,$$

or what is the same thing, if for proper values of λ, μ, ν, ρ we have

$$\lambda P_1+\mu P_2+\nu P_3+\rho P_4 = 0, \quad \lambda+\mu+\nu+\rho = 0; \text{ that is, if}$$

$$\lambda+\mu+\nu+\rho = 0, \quad \lambda\alpha_1+\mu\alpha_2+\nu\alpha_3+\rho\alpha_4 = 0, \quad \lambda\beta_1+\mu\beta_2+\nu\beta_3+\rho\beta_4 = 0,$$
$$\lambda(c_1+\gamma_1)+\mu(c_2+\gamma_2)+\nu(c_3+\gamma_3)+\rho(c_4+\gamma_4) = 0,$$
$$\lambda c_1\gamma_1+\mu c_2\gamma_2+\nu c_3\gamma_3+\rho c_4\gamma_4 = 0;$$

and eliminating from these equations $(\lambda, \mu, \nu, \rho)$, we find the above mentioned relation between $\alpha_1, \beta_1, \gamma_1, c_1$, &c.; which proves the theorem.

2. The corresponding theorem *in plano* is, that the three circles having for their respective diameters the diagonals of any quadrilateral, have a common radical axis. This may be proved by a similar analysis, or somewhat differently, in part geometrically, as follows:—Consider the circle having for the extremities of a diameter the points (α, β), (a, b) respectively; the equation of the circle is $(x-a)(x-\alpha)+(y-b)(y-\beta) = 0$. Hence, taking the three circles which correspond to the points (α_1, β_1), (a_1, b_1); (α_2, β_2), (a_2, b_2); (α_3, β_3), (a_3, b_3), these will have a common radical axis, if only

$$\lambda+\mu+\nu = 0, \quad \lambda(a_1+\alpha_1)+\mu(a_2+\alpha_2)+\nu(a_3+\alpha_3) = 0,$$
$$\lambda(b_1+\beta_1)+\mu(b_2+\beta_2)+\nu(b_3+\beta_3) = 0,$$
$$\lambda(a_1\alpha_1+b_1\beta_1)+\mu(a_2\alpha_2+b_2\beta_2)+\nu(a_3\alpha_3+b_3\beta_3) = 0;$$

or, what is the same thing, introducing a new arbitrary quantity σ, if only

$$\lambda+\mu+\nu = 0, \quad \lambda(a_1+\alpha_1)+\mu(a_2+\alpha_2)+\nu(a_3+\alpha_3) = 0,$$
$$\lambda(b_1+\beta_1)+\mu(b_2+\beta_2)+\nu(b_3+\beta_3) = 0,$$
$$\lambda a_1\alpha_1+\mu a_2\alpha_2+\nu a_3\alpha_3 = \sigma, \quad \lambda b_1\beta_1+\mu b_2\beta_2+\nu b_3\beta_3 = -\sigma.$$

Hence we have *first* $$\left|\begin{array}{ccc} 1, & 1, & 1 \\ a_1+\alpha_1, & a_2+\alpha_2, & a_3+\alpha_3 \\ b_1+\beta_1, & b_2+\beta_2, & b_3+\beta_3 \end{array}\right| = 0;$$

and then, this equation being satisfied, we have simultaneously

$$(\lambda, \mu, \nu) = \theta \left\| \begin{matrix} 1, & 1, & 1 \\ a_1+\alpha_1, & a_2+\alpha_2, & a_3+\alpha_3 \end{matrix} \right\| = \phi \left\| \begin{matrix} 1, & 1, & 1 \\ b_1+\beta_1, & b_2+\beta_2, & b_3+\beta_3 \end{matrix} \right\|,$$

where of course the value of $(\theta : \phi)$ can be found by comparing the two values of λ, μ, or ν; thus, comparing the values of λ, we have

$$\theta (a_3+\alpha_3-a_2-\alpha_2) = \phi (b_3+\beta_3-b_2-\beta_2).$$

Substituting the first set of values in the fourth equation, and the second set in the fifth equation, and adding, we find *secondly*

$$\theta \begin{vmatrix} 1, & 1, & 1 \\ a_1+\alpha_1, & a_2+\alpha_2, & a_3+\alpha_3 \\ a_1\alpha_1, & a_2\alpha_2, & a_3\alpha_3 \end{vmatrix} + \phi \begin{vmatrix} 1, & 1, & 1 \\ b_1+\beta_1, & b_2+\beta_2, & b_3+\beta_3 \\ b_1\beta_1, & b_2\beta_2, & b_3\beta_3 \end{vmatrix} = 0,$$

where $\theta : \phi$ is a given ratio.

Now if (α_1, β_1), (a_1, b_1), (α_2, β_2), (a_2, b_2), (α_3, β_3), (a_3, b_3) are the six angles of a quadrilateral; then, *first*, by a known theorem, the middle points of the three diagonals are in a line; that is, we have

$$\begin{vmatrix} 1, & 1, & 1 \\ a_1+\alpha_1, & a_2+\alpha_2, & a_3+\alpha_3 \\ b_1+\beta_1, & b_2+\beta_2, & b_3+\beta_3 \end{vmatrix} = 0;$$

and, *secondly*, by the theorem that any pencil of lines drawn to the angles of a quadrilateral are a pencil in involution, applying this to the pencils of lines parallel to the two axes respectively, we have

$$\begin{vmatrix} 1, & 1, & 1 \\ a_1+\alpha_1, & a_2+\alpha_2, & a_3+\alpha_3 \\ a_1\alpha_1, & a_2\alpha_2, & a_3\alpha_3 \end{vmatrix} = 0, \qquad \begin{vmatrix} 1, & 1, & 1 \\ b_1+\beta_1, & b_2+\beta_2, & b_3+\beta_3 \\ b_1\beta_1, & b_2\beta_2, & b_3\beta_3 \end{vmatrix} = 0.$$

The above mentioned first and second conditions are thus satisfied when the six points are the angles of a quadrilateral.

1780. (Proposed by A. Renshaw.)—Find the locus of a point whose polar, in reference to a given conic, cuts off a given area from the axes.

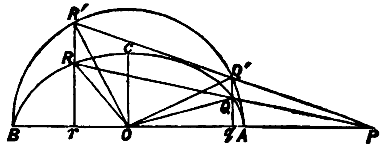
Solution by W. H. Laverty; E. McCormick; R. Tucker, M.A.; E. Fitzgerald; *the* Proposer; *and others.*

Let $Lx+My+R=0$, and $L x'+My'+R=0$, be the equations to a conic and to the polar of the point (x', y') with respect to that conic, where $(L=0)$, $(M=0)$ are the equations to the diameters bisecting chords parallel to the axes of x and y respectively, and $(R=0)$ is the polar of the origin. We may put the equation of the polar of (x', y') into the form $L'x+M'y+R'=0$, and we have $-\dfrac{R'}{L'}$, $-\dfrac{R'}{M'}$, as the intercepts on the axes; therefore $\dfrac{R'^2}{L'M'} = \mu$ (= a constant), or, omitting the accents, the equation of the required locus is $R^2 = \mu \, . \, LM$, which is that of a conic touching the diameters $(L=0)$, and $(M=0)$ of the original conic, at the points where $(R=0)$ meets those diameters.

1781. (Proposed by E. CONOLLY.)—From a point P in the produced major axis of an ellipse to draw a line PQR, cutting the curve in Q and R, so that if Q and R be joined with the centre O of the ellipse, the triangle OQR may be a maximum.

Solution by M. COLLINS, B.A.; W. H. LAVERTY; H. M. TAYLOR, B.A.; E. FITZGERALD; *and many others.*

Draw Qq, Rr perpendicular to the major axis AB, meeting the semicircle on AB in Q', R'; then, OC being the minor semi-axis, $R'r : Rr = OA : OC = Q'q : Qq$; hence $R'Q'$ passes through P;



$$\therefore \ OA : OC = \triangle POR' : POR = \triangle POQ' : POQ = \triangle OQ'R' : OQR;$$

hence the triangle OQR is greatest when $OQ'R'$ is greatest, which will plainly be the case when $Q'OR'$ is a right angle, or when $OQ'R'$ and $OR'Q'$ are each half a right angle; therefore the circle drawn round the square whose side is OP will cut the semi-circle $AQ'R'B$ in the point Q' (or R'), from which drawing $Q'q$ (or $R'r$) perpendicular to AB, it will meet the given ellipse in Q (or R) which determines the required position of PQR.

1786. (Proposed by Dr. BOOTH, F.R.S.)—If $4a$ be the parameter of a parabola, S_y the length (measured from the vertex) of an arc the ordinate of whose extremity is y, and R_y the radius of curvature at the extremity, prove that $S_{4a} - 3S_a = R_a$.

Solution by R. WARREN, B.A.; E. FITZGERALD; M. COLLINS, B.A.; E. MCCORMICK; *and many others.*

By well-known formulæ, we have

$$R_y = \frac{(y^2+4a^2)^{\frac{3}{2}}}{4a^2}, \quad S_y = \frac{y\,(y^2+4a^2)^{\frac{1}{2}}}{4a} + a\log\frac{y+(y^2+4a^2)^{\frac{1}{2}}}{2a};$$

therefore $\quad R_a = \frac{5}{4}a\sqrt{5}, \quad S_a = \frac{1}{4}a\sqrt{5} + a\log\left\{\frac{1}{2}(1+\sqrt{5})\right\},$

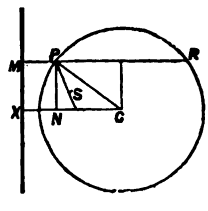
$$S_{4a} = 2a\sqrt{5} + a\log(2+\sqrt{5}) = 2a\sqrt{5} + 3a\log\left\{\tfrac{1}{2}(1+\sqrt{5})\right\};$$

therefore $$S_{4a} - 3S_a = \tfrac{5}{4}a\sqrt{5} = R_a.$$

1771. (Proposed by Professor CAYLEY.)—Given a circle and a line, it is required to find a parabola, having the line for its directrix, and the circle for a circle of curvature.

Solution by H. M. TAYLOR, B.A.; F. D. THOMSON, M.A.; M. COLLINS, B.A.; R. TUCKER, M.A.; *and others.*

From the centre C of the given circle draw CX perpendicular to the given directrix. Trisect CX, and let N be one of the points of trisection; so that CN = 2NX. Draw NP at right angles to CX, meeting the circle in P; through P draw MPR parallel to CX, meeting the circle again in R; also draw PS making an angle SPC = RPC, and take PS = $\frac{1}{4}$PR. Then a parabola, with focus S and given directrix, will plainly touch the circle at P; and will have it as a circle of curvature, since in a parabola the chord of curvature parallel to the axis is equal to 4SP.



II. *Solution by the* PROPOSER.

Let $x^2+y^2-1=0$ be the equation of the given circle, $x=m$ that of the given line. Taking on the circle an arbitrary point ($\cos\theta$, $\sin\theta$), we may find a parabola having the given line for its directrix, and touching the circle at the last mentioned point; viz., the equation of the parabola is found to be

$$y^2-2y\sin\theta\,(1+2\cos^2\theta-2m\cos\theta)-4x\cos^2\theta\,(\cos\theta-m)$$
$$+1+3\cos^2\theta-4m\cos\theta=0.$$

[There is no difficulty in *verifying* that this parabola has for its directrix the line $x-m=0$, that the equation is satisfied by the values $x=\cos\theta$, $y=\sin\theta$, and that the derived equation is satisfied by the values

$$x=\cos\theta,\ \ y=\sin\theta,\ \ \frac{dy}{dx}=-\cot\theta].$$

Representing for a moment the left-hand side of the equation by U, we have identically

$$U-\cos^2\theta\,(x^2+y^2-1)$$
$$=y^2\sin^2\theta-x^2\cos^2\theta-2y\sin\theta\,(1+2\cos^2\theta-2m\cos\theta)-4x\cos^2\theta(\cos\theta-m)$$
$$+1+4\cos^2\theta-4m\cos\theta$$
$$=(y\sin\theta+x\cos\theta-1)\,(y\sin\theta-x\cos\theta-1-4\cos^2\theta+4m\cos\theta).$$

Hence to find the intersections of the parabola with the circle, we have first

$$x^2+y^2-1=0,\ \ y\sin\theta+x\cos\theta-1=0,$$

giving the point ($\cos\theta$, $\sin\theta$) *twice*, since $y\sin\theta+x\cos\theta-1=0$ is the equation of the tangent to the circle at the point in question; and secondly

$$x^2+y^2-1=0,\ \ y\sin\theta-x\cos\theta-1-4\cos^2\theta+4m\cos\theta=0,$$

giving the remaining two points of intersection. If the circle be a circle of curvature, one of these must coincide with the point ($\cos\theta$, $\sin\theta$), that is the equation $y\sin\theta-x\cos\theta-1-4\cos^2\theta+4m\cos\theta=0$, must be satisfied by the values $x=\cos\theta$, $y=\sin\theta$; this will be the case if

$-6\cos^2\theta+4m\cos\theta=0$, that is $\cos\theta=0$, giving for the parabola $y^2\pm 2y+1=0$, which is not a proper Solution, or else $\cos\theta=\dfrac{2}{3}$, giving $\sin\theta=\pm\left(1-\dfrac{4}{9}m^2\right)^{\frac{1}{2}}$, so that there are *two* parabolas satisfying the conditions of the problem; if to fix the ideas we take the upper sign, the equation of the corresponding parabola is

$$y^2-2\left(1-\frac{4}{9}m^2\right)^{\frac{3}{2}}y+\frac{16}{27}m^3x+1-\frac{4}{3}m^2=0;$$

and it may be added that the coordinates of the focus are

$$x=m-\frac{8}{27}m^3,\quad y=\left(1-\frac{4}{9}m^2\right)^{\frac{3}{2}}.$$

The equation of the other parabola and the coordinates of the focus are of course found by merely changing the sign of the radical. The parabolas are real if $m<\frac{3}{2}$; if $m=\frac{3}{2}$ we have a single parabola, the point of contact being in this case the vertex of the parabola; and if $m>\frac{3}{2}$ the parabolas are imaginary.

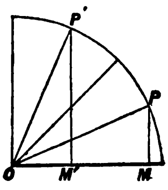
[Professor Cayley states that he was led to the foregoing problem by the consideration of the curve (proposed for investigation in Quest. 1812) which is the envelope of a variable circle having its centre in the given circle and touching the given line. The required curve (which is of the sixth order) has two cusps which, it is easy to see geometrically, are the foci of the parabolas in the problem. Taking ($\cos\theta$, $\sin\theta$) for the coordinates of the centre of the variable circle, we shall have

$$x=\tfrac{3}{2}\cos\theta-m\cos 2\theta+\tfrac{1}{2}\cos 3\theta,\; y=\tfrac{3}{2}\sin\theta-m\sin 2\theta+\tfrac{1}{2}\sin 3\theta,$$

for the coordinates of a point on the envelope.]

1773. (Proposed by M. W. Crofton, B.A.)—A traveller starts from a point on a straight river, and travels in a random direction a certain distance in a straight line. Having quite lost his way, he starts again at random the next morning, and travels the same distance; what is the chance of his reaching the river again in the second day's journey?

Solution (1) *by* H. M. Taylor, B.A.; (2) *by* E. McCormick; H. Murphy; M. Collins, B.A.; E. Fitzgerald; *the* Proposer; *and many others.*



1. Let OM be the side of the river, O the point from which the man starts, P his position at any point in the circle whose radius is the length of a day's journey. Then the chance of his reaching the river, supposing him to be at P, is $\dfrac{\angle OPM}{2\text{ rt. angles}}$; and, supposing him to be at P′, the chance of reaching the river is $\dfrac{\angle OP'M'}{2\text{ rt. angles}}$. Now let OPM + OP′M′ = one rt. angle; then the chance,

on the supposition that he is at either P or P′, is $\frac{1 \text{ rt. angle}}{4 \text{ rt. angles}}$, or $\frac{1}{4}$.

This is the chance required; since the number of ways we may choose pairs of positions P and P′ cannot affect the result, as all these pairs are equally likely to occur; P, P′ being equidistant from a line through O making half a right angle with OM.

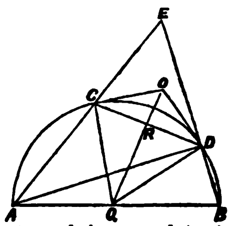
2. *Otherwise:* Let OP (in the figure to the first Solution) be the first day's journey, and PM a perpendicular to the river's bank OM; then the probability that the inclination of OP to OM lies between θ and $\theta + \Delta\theta$ is clearly $\frac{\Delta\theta}{\pi}$; and the probability of reaching the river, from P, on the second day, is $\frac{2\angle OPM}{4 \text{ rt. angles}}$ or $\frac{\pi - 2\theta}{2\pi}$; hence the whole probability required is

$$\Sigma\left(\frac{\pi-2\theta}{2\pi}\cdot\frac{\Delta\theta}{\pi}\right), \text{ or } 2\int_0^{\frac{1}{2}\pi}\frac{\pi-2\theta}{2\pi}\,\frac{d\theta}{\pi} = \frac{1}{\pi^2}\left(\pi\theta-\theta^2\right)_{\theta=0}^{\theta=\frac{1}{2}\pi} = \frac{1}{4}.$$

1774. (Proposed by H. MURPHY.)—If the ends of a fixed diameter of a circle be joined to the ends of a chord variable in position, but of given magnitude, the locus of the centre of the circle circumscribing the triangle formed by the chord and the segments of the joining lines outside the circle, is a circle concentric with the given one.

Solution by G. O. HANLON; E. CONOLLY; W. H. LAVERTY; E. MCCORMICK; R. TUCKER, M.A.; E. FITZGERALD; *the* PROPOSER; *and many others.*

Let Q be the centre of the given circle, AB the fixed diameter, CD the chord given in magnitude, E the intersection of AC, BD, and O the centre of the circle drawn round the triangle CDE. Then since the angles COR, CED are equal, their complements OCD, CAD are also equal; hence OC, OD are tangents to the given circle; therefore the quadrilateral OCQD, right-angled at C and D, is invariable in form, and the theorem is obvious.



[If AB be *any fixed chord*, the angle E is constant, being equal to the difference between the constant angles BDA and DAC; hence the locus of E is a circular segment passing through A and B.]

1060. (Proposed by EXHUMATUS.)—Two coins are thrown at random on a round table. What is the chance that they will lie upon a *diameter* of the table?

Solution by the EDITOR.

Let O be the centre of the table, r its radius; a, b the radii of the coins; 2α, 2β the angles subtended at O by the coins (a), (b), when they lie on the table at their greatest possible distance from O; also let 2θ be the angle subtended at O by the coin (a) when its centre is at a distance x from O.

We shall regard the coins as constrained to lie *wholly* on the table (as would be the case, for instance, if there were a rim round it); the calculation would of course be the same if we supposed the coins free to lie partly over the edge of the table.

Put $r-a = h$, $r-b = k$; then h, k are the radii of the circles, around O as centre, within which the centres of the coins (a), (b) must lie; and thus the probability that the centre of (a) will lie between the distances x and $x+\Delta x$, from O, is the ratio of the circular ring whose bounding radii are $(x, x+\Delta x)$ to the whole circle (h), that is $2\pi x\Delta x : \pi h^2$, or $2x\Delta x : h^2$. Through O draw two tangents (QOQ′, ROR′, which include an angle QOR $= 2\theta$) to the circle (a); within the angle QOR′ draw two lines (oq, or') parallel to OQ, OR′, and at a distance b from them; draw also the like lines $(o'q', o'r)$ within the angle Q′OR; and let these parallels meet the circle (k) in q, r', q', r. Then it is clear that some part of both coins will lie on the same diameter of the table when (and *only* when) the centre of (b) falls anywhere on the area $qor'q'or$ (= S suppose).

Now S is made up of four sectors whose angle is $(\theta+\beta)$, and four triangles whose altitude is b and base $b(\cot\beta - \tan\theta)$, hence we have

$$\mathrm{S} = 2k^2(\theta+\beta) + 2b^2(\cot\beta - \tan\theta).$$

Moreover when $\theta > \frac{1}{2}\pi - \beta$, or $x < a\sec\beta$, both coins *must* lie on a diameter of the table. Hence the whole probability (p) required is

$$p = \int_0^{a\sec\beta} \frac{2xdx}{h^2} + \int_{a\sec\beta}^{h} \frac{\mathrm{S}}{\pi k^2}\cdot\frac{2xdx}{h^2} = \frac{a^2}{h^2}\sec^2\beta + \frac{2a^2}{\pi h^2}\int(\theta+\beta)\,.\,d(\operatorname{cosec}^2\theta)$$

$$+ \frac{2a^2b^2}{\pi h^2k^2}\int\left\{\cot\beta\,.\,d(\operatorname{cosec}^2\theta) + 2\operatorname{cosec}^2\theta d\theta\right\}$$

$$= \frac{a^2}{h^2}\sec^2\beta + \frac{2a^2}{\pi h^2}\left\{(\theta+\beta)\operatorname{cosec}^2\theta + \cot\theta\right\} + \frac{2a^2b^2}{\pi h^2k^2}\left\{\cot\beta\operatorname{cosec}^2\theta - 2\cot\theta\right\}.$$

Taking these integrals between limits $(\theta = \frac{1}{2}\pi - \beta,\ \theta = \alpha)$, and putting for h, k their values $a\operatorname{cosec}\alpha$, $b\operatorname{cosec}\beta$, we obtain finally

$$p = \frac{1}{\pi}\left\{(2\alpha+2\beta) + \sin(2\alpha+2\beta)\right\}.$$

As a *numerical* example, let it be required to find the probability that two florins (diameters $1\frac{1}{4}$ inches, suppose,) thrown on a table 50 inches in diameter, will lie on a diameter of the table.

$$\text{Here} \quad \alpha = \beta = \sin^{-1}\left(\tfrac{1}{39}\right) = 1^\circ\ 28'\ 10'';$$

$$\therefore\ p = \frac{1}{\pi}(4\alpha + \sin 4\alpha) = \cdot 06524 = \tfrac{1}{15} \text{ nearly, or } \tfrac{3}{46} \text{ more nearly, \&c.}$$

Hence the odds are about 14 to 1 *against* their resting on a diameter of the table.

Solution by PROFESSOR CAYLEY; *and* H. M. TAYLOR, B.A.

1. Let the given line be taken for the axis of z; the axes of x, y being any rectangular axes in the plane perpendicular thereto; the equations of the given line are therefore ($x=0$, $y=0$). Take $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$, $(\alpha_4, \beta_4, \gamma_4)$ for the coordinates of the points A, B, C, D respectively; and $(0, 0, c_1)$, $(0, 0, c_2)$, $(0, 0, c_3)$, $(0, 0, c_4)$ for the coordinates of the points a, b, c, d respectively. Then, to determine c_1, the equation of the plane BCD is

$$\begin{vmatrix} x, & y, & z, & 1 \\ \alpha_2, & \beta_2, & \gamma_2, & 1 \\ \alpha_3, & \beta_3, & \gamma_3, & 1 \\ \alpha_4, & \beta_4, & \gamma_4, & 1 \end{vmatrix} = 0,$$

and cutting this by the line $x=0$, $y=0$, we have

$$\begin{vmatrix} 0, & 0, & c_1, & 1 \\ \alpha_2, & \beta_2, & \gamma_2, & 1 \\ \alpha_3, & \beta_3, & \gamma_3, & 1 \\ \alpha_4, & \beta_4, & \gamma_4, & 1 \end{vmatrix} = 0,$$

with similar equations for c_2, c_3, c_4 respectively. The four equations may be united into the single equation

$$\begin{vmatrix} c_1p_1, & 1, & \alpha_1, & \beta_1 \\ c_2p_2, & 1, & \alpha_2, & \beta_2 \\ c_3p_3, & 1, & \alpha_3, & \beta_3 \\ c_4p_4, & 1, & \alpha_4, & \beta_4 \end{vmatrix} = \begin{vmatrix} p_1, & \alpha_1, & \beta_1, & \gamma_1 \\ p_2, & \alpha_2, & \beta_2, & \gamma_2 \\ p_3, & \alpha_3, & \beta_3, & \gamma_3 \\ p_4, & \alpha_4, & \beta_4, & \gamma_4 \end{vmatrix},$$

where p_1, p_2, p_3, p_4 are arbitrary multipliers. Hence, writing successively $(p_1, p_2, p_3, p_4) = (1, 1, 1, 1)$ and $(p_1, p_2, p_3, p_4) = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$; we have *first*

$$\begin{vmatrix} c_1, & 1, & \alpha_1, & \beta_1 \\ c_2, & 1, & \alpha_2, & \beta_2 \\ c_3, & 1, & \alpha_3, & \beta_3 \\ c_4, & 1, & \alpha_4, & \beta_4 \end{vmatrix} = \begin{vmatrix} 1, & \alpha_1, & \beta_1, & \gamma_1 \\ 1, & \alpha_2, & \beta_2, & \gamma_2 \\ 1, & \alpha_3, & \beta_3, & \gamma_3 \\ 1, & \alpha_4, & \beta_4, & \gamma_4 \end{vmatrix},$$

that is,

$$\begin{vmatrix} 1, & \alpha_1, & \beta_1, & c_1+\gamma_1 \\ 1, & \alpha_2, & \beta_2, & c_2+\gamma_2 \\ 1, & \alpha_3, & \beta_3, & c_3+\gamma_3 \\ 1, & \alpha_4, & \beta_4, & c_4+\gamma_4 \end{vmatrix} = 0;$$

and *secondly*,

$$\begin{vmatrix} c_1\gamma_1, & 1, & \alpha_1, & \beta_1 \\ c_2\gamma_2, & 1, & \alpha_2, & \beta_2 \\ c_3\gamma_3, & 1, & \alpha_3, & \beta_3 \\ c_4\gamma_4, & 1, & \alpha_4, & \beta_4 \end{vmatrix} = \begin{vmatrix} \gamma_1, & \alpha_1, & \beta_1, & \gamma_1 \\ \gamma_2, & \alpha_2, & \beta_2, & \gamma_2 \\ \gamma_3, & \alpha_3, & \beta_3, & \gamma_3 \\ \gamma_4, & \alpha_4, & \beta_4, & \gamma_4 \end{vmatrix},$$

that is,

$$\begin{vmatrix} 1, & \alpha_1, & \beta_1, & c_1\gamma_1 \\ 1, & \alpha_2, & \beta_2, & c_2\gamma_2 \\ 1, & \alpha_3, & \beta_3, & c_3\gamma_3 \\ 1, & \alpha_4, & \beta_4, & c_4\gamma_4 \end{vmatrix} = 0;$$

and these two results may be united into the single formula

$$\left\|\begin{matrix} 1, & \alpha_1, & \beta_1, & c_1+\gamma_1, & c_1\gamma_1 \\ 1, & \alpha_2, & \beta_2, & c_2+\gamma_2, & c_2\gamma_2 \\ 1, & \alpha_3, & \beta_3, & c_3+\gamma_3, & c_3\gamma_3 \\ 1, & \alpha_4, & \beta_4, & c_4+\gamma_4, & c_4\gamma_4 \end{matrix}\right\| = 0.$$

Now the equation of a sphere having for the extremities of a diameter the points (α, β, γ) and (a, b, c) is

$$[x-\tfrac{1}{2}(a+\alpha)]^2+[y-\tfrac{1}{2}(b+\beta)]^2+[z-\tfrac{1}{2}(c+\gamma)]^2 = \tfrac{1}{4}[(a-\alpha)^2+(b-\beta)^2+(c-\gamma)^2],$$

or, $$(x-a)(x-\alpha)+(y-b)(y-\beta)+(z-c)(z-\gamma) = 0,$$

or, $$x^2+y^2+z^2-(a+\alpha)x-(b+\beta)y-(c+\gamma)z+a\alpha+b\beta+c\gamma = 0;$$

therefore, when the two points are (α, β, γ) and $(0, 0, c)$, the equation is

$$x^2+y^2+z^2-\alpha x-\beta y-(c+\gamma)z+c\gamma = 0.$$

Hence, putting for shortness $P = -\alpha x-\beta y-(c+\gamma)z+c\gamma$, viz., $P_1 = -\alpha_1 x-\beta_1 y-(c_1+\gamma_1)z+c_1\gamma_1$, &c., the equations of the four spheres are

$$x^2+y^2+z^2+P_1 = 0, \quad x^2+y^2+z^2+P_2 = 0, \quad x^2+y^2+z^2+P_3 = 0,$$
$$x^2+y^2+z^2+P_4 = 0,$$

and the four spheres will have a common radical axis, if for proper values of the multipliers μ, ν, ρ we have

$$\mu(P_1-P_2)+\nu(P_1-P_3)+\rho(P_1-P_4) = 0,$$

or what is the same thing, if for proper values of λ, μ, ν, ρ we have

$$\lambda P_1+\mu P_2+\nu P_3+\rho P_4 = 0, \quad \lambda+\mu+\nu+\rho = 0; \text{ that is, if}$$

$$\lambda+\mu+\nu+\rho = 0, \quad \lambda\alpha_1+\mu\alpha_2+\nu\alpha_3+\rho\alpha_4 = 0, \quad \lambda\beta_1+\mu\beta_2+\nu\beta_3+\rho\beta_4 = 0,$$
$$\lambda(c_1+\gamma_1)+\mu(c_2+\gamma_2)+\nu(c_3+\gamma_3)+\rho(c_4+\gamma_4) = 0,$$
$$\lambda c_1\gamma_1+\mu c_2\gamma_2+\nu c_3\gamma_3+\rho c_4\gamma_4 = 0;$$

and eliminating from these equations $(\lambda, \mu, \nu, \rho)$, we find the above mentioned relation between $\alpha_1, \beta_1, \gamma_1, c_1$, &c.; which proves the theorem.

2. The corresponding theorem *in plano* is, that the three circles having for their respective diameters the diagonals of any quadrilateral, have a common radical axis. This may be proved by a similar analysis, or somewhat differently, in part geometrically, as follows:—Consider the circle having for the extremities of a diameter the points (α, β), (a, b) respectively; the equation of the circle is $(x-a)(x-\alpha)+(y-b)(y-\beta) = 0$. Hence, taking the three circles which correspond to the points (α_1, β_1), (a_1, b_1); (α_2, β_2), (a_2, b_2); (α_3, β_3), (a_3, b_3), these will have a common radical axis, if only

$$\lambda+\mu+\nu = 0, \quad \lambda(a_1+\alpha_1)+\mu(a_2+\alpha_2)+\nu(a_3+\alpha_3) = 0,$$
$$\lambda(b_1+\beta_1)+\mu(b_2+\beta_2)+\nu(b_3+\beta_3) = 0,$$
$$\lambda(a_1\alpha_1+b_1\beta_1)+\mu(a_2\alpha_2+b_2\beta_2)+\nu(a_3\alpha_3+b_3\beta_3) = 0;$$

or, what is the same thing, introducing a new arbitrary quantity σ, if only

$$\lambda+\mu+\nu = 0, \quad \lambda(a_1+\alpha_1)+\mu(a_2+\alpha_2)+\nu(a_3+\alpha_3) = 0,$$
$$\lambda(b_1+\beta_1)+\mu(b_2+\beta_2)+\nu(b_3+\beta_3) = 0,$$
$$\lambda a_1\alpha_1+\mu a_2\alpha_2+\nu a_3\alpha_3 = \sigma, \quad \lambda b_1\beta_1+\mu b_2\beta_2+\nu b_3\beta_3 = -\sigma.$$

Hence we have *first*

$$\begin{vmatrix} 1\ , & 1\ , & 1 \\ a_1+\alpha_1, & a_2+\alpha_2, & a_3+\alpha_3 \\ b_1+\beta_1, & b_2+\beta_2, & b_3+\beta_3 \end{vmatrix} = 0;$$

7. In order to find the envelope of the line $\alpha\beta\gamma$, given by equation (i), we have, if u, v, w be the coefficients of L, R, M respectively,

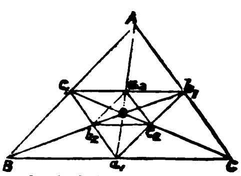
$$\frac{u}{2\mu abc} = \frac{v}{\mu^3 + \mu^2(a+b+c) - \mu(ab+bc+ca) - abc} = \frac{w}{-2\mu^2};$$

$$\therefore\ w^3abc - w^2u(a+b+c) - uw^2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + \frac{u^3}{abc} = 2uvw \ldots\ldots \text{(vi)},$$

a *tangential* equation to the envelope, which is therefore a curve of the third class touching the line $w = 0$, $u = 0$, that is, the line R. Now the coordinates of the line BC are proportional to bc, $-(b+c)$, 1, and it will be found that they satisfy equation (vi). Hence, the envelope touches the sides of the given triangle. Again, the coordinates of CR will be seen to be proportional to abc, $c^2 - ab$, $-c$, which satisfy (vi); therefore, *in the projection*, the envelope (vi) touches the line at infinity, the three sides of the triangle and the three perpendiculars. See the Solutions of Questions 1679, 1680. (*Reprint*, Vol. III., pp. 81—84.)

1727. (Proposed by E. Conolly.)—If the middle points of the sides of any triangle be joined, and the middle points of the sides of this new triangle also joined, and so on *ad infinitum*; the limit to which these triangles tend is a point, and the sum of the squares on the lines drawn therefrom to the angles of all the inscribed triangles is one-third of the sum of the squares on the lines drawn from the same point to the angles of the original triangle.

Solution by J. Dale; W. H. Laverty; *the* Proposer; *and others.*



It is evident from the construction that the lines Aa_1, Bb_1, Cc_1, which bisect BC, CA, AB respectively, also bisect the sides of all the inscribed triangles, which are respectively parallel to BC, CA, AB; and hence it follows that the ultimate limit of the inscribed triangles is an infinitesimal triangle, having its sides proporportional to BC, CA, AB and the point O for its *centroid* (or centre of gravity). At the limit, the infinitesimal triangle and the point O may be supposed, without sensible error, to coincide. Now, by a well-known theorem, we have $Oa_1 = \frac{1}{2}OA$, $Oa_2 = \frac{1}{2}Oa_1 = \frac{1}{4}OA$, $Oa_3 = \frac{1}{8}OA$, and so on *ad infinitum*;

therefore $\qquad \Sigma Oa_1^2 = OA^2\left(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots\right) = \frac{1}{3}OA^2;$

similarly $\qquad \Sigma Ob_1^2 = \frac{1}{3}OB^2$, and $\Sigma Oc_1^2 = \frac{1}{3}OC^2;$

therefore $\qquad \Sigma Oa_1^2 + \Sigma Ob_1^2 + \Sigma Oc_1^2 = \frac{1}{3}(OA^2 + OB^2 + OC^2).$

1829. (Proposed by M. W. Crofton, B.A.)—Two points being taken at random within (1) a circle or (2) a sphere, what is the probability that their distance apart is less than a given line?

Solution by Professor Sylvester.

1. *For the circle.* Call the radius 1, the length of the given line $2k$, the measure of the favourable cases F, and the required probability p.

(α). Block out from the given circle the interior concentric circle πk^2.

(β). Resolve the combinations which supply the remaining favourable cases into concentric rings, each of thickness dr, enclosing circles of radii varying from k to r, which must be *doubled* in order that each ring may be brought into combination with each that precedes as well as with each that follows it. Each such ring may be regarded as a circumference loaded with a uniform density $2\pi r dr$.

(γ). From any one of the circles thus enclosed block out a smaller circle πk^2, and resolve the remainder into rings of variable thickness, by drawing circles of all radii from k to r touching one another at the same point.

(δ). Divide each such ring into quadrilateral elements by drawing normals to the bounding contours all round.

The first circle blocked out supplies to F a portion $\pi^2 k^4$. The sets of circles subsequently blocked out supply a portion represented by

$$\int_k^1 4\pi r dr\, (\pi k^2) = 2\pi^2 k^2 - 2\pi^2 k^4.$$

The combinations remaining supply to M a portion represented by

$$\int_k^1 4\pi r dr \int_k^r \frac{dr}{r}\, \Sigma\, (pds),$$

where s denotes the arc of a circle to radius r extending from arc-chord $2k$ in one direction to arc-chord $2k$ in the contrary direction, and p the perpendicular from the origin of s on the direction of ds. For, calling n the element of the normal (drawn at the extremity of s) intercepted between two circular contours of radii r, $r+dr$, respectively, and remembering that the point of contact between these two circles is their centre of similitude, we have $\frac{n}{p}$ constant and consequently $= \frac{d\,(2r)}{2r} = \frac{dr}{r}$, so that $\Sigma\,(nds) = \frac{dr}{r}\,\Sigma\,(pds)$. Thus, in fact, if A is the area of a segment, to radius r, whose chord is $2k$, we have

$$\Sigma\,(pds) = 2\,(2\mathrm{A}) = 2\int_0^k \frac{(2\rho)^2\, d\rho}{\sqrt{(r^2-\rho^2)}} = 8\int_0^k \frac{\rho^2 d\rho}{\sqrt{(r^2-\rho^2)}};$$

and if we write $\rho = ru$, and then make $\frac{k}{r} = t$, and finally replace under the signs of summation u and t each by k, we obtain

$$\mathrm{F} - 2\pi^2 k^2 + \pi^2 k^4 = 32\pi k^4 \int_1^k \frac{dk}{k^3} \int_1^k \frac{dk}{k^3} \left\{ \int_1^k \frac{k^2 dk}{\sqrt{(1-k^2)}} + \int_0^1 \frac{k^2 dk}{\sqrt{(1-k^2)}} \right\}$$

$$= 4\pi k^4 \left\{ \frac{1}{k^4}\int_1^k \frac{k^2 dk}{\sqrt{(1-k^2)}} - \frac{2}{k^2}\int_1^k \frac{dk}{\sqrt{(1-k^2)}} + \int_1^k \frac{dk}{k^2\sqrt{(1-k^2)}} \right\} + \pi^2(1-k^2)^2$$

$$= 2\pi \left\{ \sin^{-1} k - k\sqrt{(1-k^2)} - \tfrac{1}{2}\pi \right\} - 8\pi k^2 (\sin^{-1} k - \tfrac{1}{2}\pi) - 4\pi k^3 \sqrt{(1-k^2)} + \pi^2 (1-k^2)^2;$$

$$\therefore\ p = \frac{F}{\pi^2} = \frac{2}{\pi}(1-4k^2)\sin^{-1} k - \frac{2}{\pi}(k+2k^3)\sqrt{(1-k^2)} + 4k^2.$$

The same result might, of course, have been obtained, but less elegantly and less concisely, by writing at once in place of A its value as a function of k and r.

2. *For the sphere.* Following precisely the same method as in (α), (β), (γ) of the preceding, but of course substituting spherical shells for rings, we have

$$F - 2(\tfrac{4}{3}\pi)(\tfrac{4}{3}\pi k^3) + (\tfrac{4}{3}\pi)^2 k^6 = \int_k^1 2(4\pi r^2 dr)\int_k^r \frac{dr}{r}(B),$$

where $B = \Sigma(pdS)$, dS being an element of a spherical surface swept out by the revolution of the arc of the circle of radius r, cut off by the chord $2k$, about a diameter passing through an extremity of the arc. Thus B is three times the volume swept out by A in this revolution; that is, h being the versed-sine of the arc, $B = \pi h^2 r = \dfrac{4\pi k^4}{r}$; hence the probability is

$$p = 2k^3 - k^6 + 18k^4 \int_k^1 r^2 dr \int_k^r \frac{dr}{r^2} = 8k^3 - 9k^4 + 2k^6.$$

NOTE.—The solution here given, although it may appear unnecessarily elaborate as regards the particular problem to be resolved, is of importance as illustrating a general method developed by the writer for dealing with questions of this sort—a method which subjectively has taken its rise out of the admirable observation made use of by Mr. Woolhouse in his solution of the problem (Quest. 1333) for determining the probability of a triangle joining three points taken at random within a sphere being acute. Mr. Woolhouse, and Mr. Crofton after him, (*Educational Times* for Jan. 1863 and Oct. 1865, or *Reprint*, Vol. IV., pp. 61, 81,) have shown that one point of the group of three may be considered as fixed in the circumference without affecting the probability in question. See a brief notice of my paper read before the British Association at Birmingham, in the *Athenæum* and *Reader* of Oct. 7th, 1865, [also a notice, unfortunately still briefer for want of space, on p. 179 of the *Educational Times* for Nov., 1865]. The reductions in the order of summations therein referred to can be increased to *three* degrees for plane rectilinear, and to *four* degrees for solid polyhedral, contours.

[If we apply the foregoing method to find the probability ($1-p$, or q) that the distance between the two points is *greater* than $2k$, we need only attend to the process in (γ) and (δ), no other terms entering into our investigation; the complementary probability in question then is, for the circle,

$$q = \frac{1}{\pi^2}\int_k^1 4\pi r\,dr \int_k^r \frac{dr}{r}\int_k^r \frac{8\rho^2 d\rho}{\sqrt{(r^2-\rho^2)}} = \frac{4k^4}{\pi}\left(\int_1^{\frac{1}{k^2}} dx\right)^3 \frac{1}{x\cdot\sqrt{(x-1)}};$$

the latter remarkably simple result being obtained from the former by putting

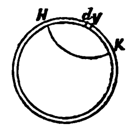
first $\rho = ru$, then $r = \frac{k}{t}$, and finally $x = \frac{1}{t^2}$. Reducing this to a series of (three) single integrals, by the aid of a known and elegant theorem for effecting such reduction of an integral of the nth order (see Boole's *Differential Equations*, 2nd ed., p. 398), the result is readily seen to be identical with that given above.]

II. *Solution by the* PROPOSER.

Let r be the given radius, a the length of the given line; and let us consider the problem, first, for the *sphere*.

If p be the probability of any event, F the number of favourable cases, W the whole number of cases; we have $p = \frac{\text{F}}{\text{W}}$; so that if F and W are determined, p is known.

We may represent the number of points within a given space, by the space itself; hence if S, S′ be two spaces, the number of pairs of points given by taking one point in each is SS′; and the number of pairs given by points within the same space S is $\frac{1}{2}\text{S}^2$. In the present question, $\text{W} = \frac{8}{9}\pi r^6$. To determine F, let r become $r + dr$; and, taking an infinitely small element $d\text{V}$ of the spherical shell so formed as centre, draw a sphere of radius a cutting the other; the *increase* in the number of favourable cases given by taking one point in the element $d\text{V}$ is (lune HK) $\times\, d\text{V}$; hence the whole increase is



$$d\text{F} = (\text{lune HK}) \times (\text{volume of spherical shell}).$$

$$\text{Now lune HK} = \frac{2\pi}{3}a^3 - \frac{\pi}{4}\frac{a^4}{r}, \therefore d\text{F} = \left(\frac{2\pi}{3}a^3 - \frac{\pi}{4}\frac{a^4}{r}\right)4\pi r^2 dr;$$

$$\therefore \text{F} = 4\pi^2 a^3 \int \left(\frac{2}{3} - \frac{a}{4r}\right) r^2 dr = 4\pi^2 a^3 \left(\frac{2}{9}r^3 - \frac{a}{8}r^2\right) + \text{C};$$

therefore
$$p = \frac{\text{F}}{\text{W}} = \frac{a^3}{r^3} - \frac{9}{16}\frac{a^4}{r^4} + \frac{\text{C}'}{r^6}.$$

To determine the arbitrary constant C′, we observe that if $a = 2r$, $p = 1$, whence $\text{C}' = \frac{1}{32}a^6$; therefore the required probability is

$$p = \frac{a^3}{r^3} - \frac{9}{16}\frac{a^4}{r^4} + \frac{1}{32}\frac{a^6}{r^6}.$$

If the distance of the points is to be less than the *radius* of the *sphere* (or $a = r$), we have $p = \frac{15}{32}$.

For a *circle* the integration is rather more difficult; I find the result to be

$$p = \frac{a^2}{r^2} + \frac{2}{\pi}\left(1 - \frac{a^2}{r^2}\right)\sin^{-1}\frac{a}{2r} - \frac{a}{4\pi r}\left(2 + \frac{a^2}{r^2}\right)\left(4 - \frac{a^2}{r^2}\right)^{\frac{1}{2}}.$$

In this case if $a = r$, we have $p = 1 - \frac{3\sqrt{3}}{4\pi}$.

NOTE ON QUESTION 1333; BY W. S. B. WOOLHOUSE, F.R.A.S.

The following summary statement is made at the suggestion of the Mathematical Editor of the *Educational Times*, and also at the request of Professor Sylvester, who has latterly become much interested in the general principles which bear upon the subject.

This problem was first proposed by me amongst the mathematical questions of the *Lady's and Gentleman's Diary* for 1861, and an able answer to it by Mr. Stephen Watson was given in the Diary for 1862. Another correct and satisfactory answer was also sent by Mr. C. H. Brooks, but it was too elaborate for insertion in that publication.

The solution printed in the *Educational Times* for January, 1863, was literally copied from my own manuscript investigation, drawn up some years before the question was proposed. In working out this solution, the principle of limiting one of the points to the surface of the sphere was really considered to be self-evident, and a sort of preliminary deduction [added at the end of this Note] was given more particularly for the purpose of adverting to the remarkable peculiarity that the acute triangles occupying the sphere and shell in the manner stated are not proportionally partitioned in the respective spaces. The determination of the probabilities p'', p''' there defined, will be found to be a curious subject of inquiry, when the shell is supposed to have a finite thickness.

So far as I am concerned as the originator of this question, I do not see much to claim in the suggestive idea of limiting one of the points to the surface; but whatever merit may be attached to it in the estimation of others, must be fully shared by Mr. Brooks, who, in his solution sent to the Diary, expressly adopts the same principle. Being myself at first under the impression that this simplifying expedient was essential to the investigation, I was somewhat pleased with the ingenuity shown by Mr. Watson, whose solution, printed in the Diary, is in this respect quite independent and unique.

Considering the problem in its more general form, whatever be the boundary of the figure, whether regular or irregular, it is obvious that one of the points may be restricted to the outer surface under certain conditions. Thus, for example, if within any proposed volume a point be taken as a centre of similitude, and the figure be conceived to receive a proportional increment from the centre in every direction, or if the augmented figure be only similar to the original one, whether similarly situated or not, it will follow that the probability will be the same if one of the points be restricted to the surface, provided that there be assigned to its existence a probability measured by the proportional increment of volume which appertains to the immediate vicinity of the point in question. In the case of the sphere, the point may not merely be restricted to the surface, but, on account of the general symmetry, it may be made a fixed point.

The most difficult phase of this question, and by far the most interesting, is that in which the three points are taken within the volume of a given parallelepiped, and to this no solution has yet been given. But my own investigation of the analogous case of a rectangle has already appeared in the Reprint of "Mathematical Questions with their Solutions, from the Educational Times," vol. i. p. 71.

In conclusion I may add that the principle by which a point is limited to a boundary may be stated in still more general terms as follows:—

If a figure be subjected to a small variation, in such a manner that two accumulative symmetrical functions of any kind, depending upon a given

number of arbitrary points, shall have to each other the same ratio in the varied as in the primitive figure, that ratio will retain the same value when one of the points is restricted, positively or negatively, to the increment or decrement of the figure, as the case may be.

As regards the proof, it is sufficient to indicate that it follows from the simple and obvious analytical deduction, that if $\frac{Q+\delta Q}{R+\delta R}=\frac{Q}{R}$, however small the increments $\delta Q, \delta R$ be taken, then will $\frac{dQ}{dR}=\frac{Q}{R}$; and this applies equally to any form of enunciation.

[The "preliminary deduction" referred to above is as follows:—

"Let V denote the volume of a sphere, the number of available points comprised within it being proportionally represented by V. Then the number of triangles that can be formed by taking points three and three, will be $\frac{V^3}{2.3}$. Suppose V to receive a finite increment, and to become $V+\delta V$; then the total number of triangles that can be formed in the increased volume will be

$$\frac{(V+\delta V)^3}{2.3}=\frac{V^3}{2.3}+\frac{V^2}{2}.\delta V+V\frac{\delta V^2}{2}+\frac{\delta V^3}{2.3}, \text{ or } N=n_0+n_1+n_2+n_3\ldots.(\alpha).$$

Here the first term n_0 is the number of triangles wholly contained in V; n_1 is the number of those which have two angular points in V and one in δV; n_2 is the number of those which have all three points within the spherical shell δV. Let p denote the required probability (of an acute triangle), then the number of *acute* triangles contained amongst N will be $pN=pn_0+pn_1+pn_2+pn_3$; but it is a peculiar feature of this question that the acute triangles are not thus locally partitioned as in (α). If p', p'', p''' be the respective probabilities under the several partitions, the values collected under those partitions will be $pN=pn_0+p'n_1+p''n_2+p'''n_3$, and p', p'', p''' will, in fact, be functions of $\frac{\delta V}{V}$.

Equating the two values of pN thus represented, we must have

$$0=(p'-p)+(p''-p)\frac{n_2}{n_1}+(p'''-p)\frac{n_3}{n_1}=(p'-p)+(p''-p)\frac{\delta V}{V}+\tfrac{1}{3}(p'''-p)\frac{\delta V^2}{V^2}.$$

If we now conceive the finite increment δV to be evanescent, this equation will become $p_0'-p=0$, or $p=p_0'$; that is, the required probability is the same as that which obtains when one of the points is limited to the surface of the given sphere."]

1790. (Proposed by Professor SYLVESTER.)—(1.) If a set of six points be respectively represented by the six permutations of $a:b:c$, show that they lie in a conic, and write down its equation.

(2.) *Hence* prove that if AB, BC, CA be three real lines containing the nine points of inflexion of a cubic curve having an oval, the pairs of tangents drawn to the oval from A, B, C will meet it in six points lying in a conic.

I. *Solution by* Professor CAYLEY.

1. That the six points,

$$1 = (a, b, c), \quad 2 = (b, c, a), \quad 3 = (c, a, b),$$
$$4 = (a, c, b), \quad 5 = (b, a, c), \quad 6 = (c, b, a),$$

are situate on a conic, appears at once by writing down its equation: viz.,

$$(bc + ca + ab)(x^2 + y^2 + z^2) - (a^2 + b^2 + c^2)(yz + zx + xy) = 0,$$

which is satisfied by the coordinates of each of the six points.

2. It is interesting to remark that the six points on the conic form, not a general inscribed hexagon, but a hexagon such as is mentioned in Prob. 1512 (*Reprint*, Vol. II., p. 51), viz., one in which the three diagonals pass respectively through the Pascalian points (intersections of opposite sides): in fact, in the hexagon 143526, forming the equations of the sides and diagonals, these are

14. $x(b + c) - ay - az = 0$, 25. $x(c + a) - by - bz = 0$,
15. $-cx - cy + (a + b)z = 0$, 26. $-ax - ay + (b + c)z = 0$,
16. $-bx + (c + a)y - bz = 0$, 24. $-cx + (a + b)y - cz = 0$,

36. $x(a + b) - cy - cz = 0$,
34. $-bx - by + (c + a)z = 0$,
35. $-ax + (b + c)y - az = 0$;

so that the lines 14, 25, 36 meet in the point $x = 0, \ y + z = 0$,
,, 16, 24, 35 ,, $y = 0, \ z + x = 0$,
,, 15, 26, 34 ,, $z = 0, \ x + y = 0$.

3. It is further to be remarked that the six points lie on the cubic curve

$$\frac{x^3 + y^3 + z^3}{a^3 + b^3 + c^3} - \frac{xyz}{abc} = 0,$$

and are consequently the six points of intersection of this cubic by the above mentioned conic.

4. The points $(x = 0, \ y + z = 0)$, $(y = 0, \ z + x = 0)$, $(z = 0, \ x + y = 0)$ are the three real inflexions of the cubic; hence, attending only to the cubic, and starting from the arbitrary point (a, b, c) on this curve, it appears by what precedes, that we may, by means of the three real inflexions of the cubic, construct the system of six points, (the construction is, in fact, identical with that given in my Solution of Problem 1744, *Reprint*, Vol. IV., pp. 32—37, the six points being six out of the therein mentioned eighteen points); and it further appears, that these six points lie on a conic.

5. As regards the second part of the proposed Problem, consider the cubic curve $x^3 + y^3 + z^3 + 6lxyz = 0$; the three real lines containing the nine points of inflexion are the lines $x = 0$, $y = 0$, $z = 0$; and the points A, B, C are therefore $(y = 0, \ z = 0)$, $(z = 0, \ x = 0)$, $(x = 0, \ y = 0)$ respectively. From each of these points we may draw to the curve six tangents, and we have thus on the curve eighteen points, which are a particular case of the system in the Solution of Prob. 1744. Or if from each of the points we draw two properly selected tangents, (when the cubic has an oval these may be the two tangents to the oval,) then we obtain a system of six points, (part of the system of eighteen points); viz., the coordinates of the six

points are of the form (a, b, c), (b, c, a), (c, a, b), (a, c, b), (b, a, c), (c, b, a) and therefore the six points are in a conic.

6. To verify this, if we take $y = \theta x$ for the equation of a tangent from the point $(x = 0, y = 0)$, the equation $(1+\theta^3)x^3 + 6l\theta x^2 z + z^3 = 0$ must have a pair of equal roots, giving for θ the equation $(1+\theta^3)^2 + 32l^3\theta^3 = 0$; and we then find $z = -\dfrac{1+\theta^3}{4l\theta}x$, that is, θ being determined by the foregoing equation, the coordinates of the point of contact are $x : y : z = 1 : \theta : -\dfrac{1+\theta^3}{4l\theta}$. The roots of the equation in θ are of the form $\theta_1, \theta_2, \theta_3, \dfrac{1}{\theta_1}, \dfrac{1}{\theta_2}, \dfrac{1}{\theta_3}$; and assuming that the curve has an oval, there are two real roots $\theta_1, \dfrac{1}{\theta_1}$. Hence, writing $x : y : z = 1 : \theta_1 : -\dfrac{1+\theta_1^3}{4l\theta_1} = a : b : c$, the substitution $\dfrac{1}{\theta_1}$ for θ, gives $x : y : z = b : a : c$, that is, the coordinates of the points of contact of the tangents to the oval, from the point $(x = 0, y = 0)$ are (a, b, c) and (b, a, c) respectively; and writing successively (y, z, x) and (z, x, y) in place of (x, y, z), the coordinates for the tangents from $(y = 0, z = 0)$ are (b, c, a), (c, b, a); and those for the tangents from $(z = 0, x = 0)$ are (c, a, b) and (a, c, b); so that the coordinates of the six points of contact are a system of the form in question.

II. *Solution by the* Proposer.

(1.) This needs no proof: it is enough to *write down* $\Sigma ab\Sigma x^2 - \Sigma a^2\Sigma yz = 0$.
(2.) Writing the equation to the cubic under the form $x^3 + y^3 + z^3 + 6lxyz = 0$, where l is real, its points of contact with the 6 tangents from $(x=0, y=0)$ will be found by combining the above with $z^2 + 2lxy = 0$, which gives

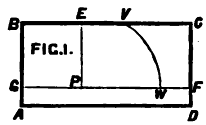
$$\frac{x^3}{z^3} + \frac{y^3}{z^3} = 2, \quad \frac{x^3}{z^3}\cdot\frac{y^3}{z^3} = -\frac{1}{8l^3}.$$

Hence $\dfrac{x^3}{z^3}, \dfrac{y^3}{z^3}$ are the roots, say ρ, ρ' of the equation $\rho^2 - 2\rho - \dfrac{1}{8l^3} = 0$; and ρ, ρ' must be real, or there could not be an oval; for the oval (when there is one) always lies entirely within or entirely without the triangle ABC. The six points of contact with the *cubic* arise from combining the three values of $\sqrt[3]{\rho}$ with the three values of $\sqrt[3]{\rho'}$, subject to the condition that $\sqrt[3]{\rho}\cdot\sqrt[3]{\rho'} = -\dfrac{1}{2l}$. Thus, using $\sqrt[3]{\rho}, \sqrt[3]{\rho'}$ to denote the real values of the cube roots, the two points of contact with the oval of the tangent from $(x=0, y=0)$ will have $1 : \sqrt[3]{\rho} : \sqrt[3]{\rho'}$ and $1 : \sqrt[3]{\rho'} : \sqrt[3]{\rho}$ for their x, y, z coordinates respectively. And proceeding in like manner with the other two pairs of tangents, we see that the six permutations of $1 : \sqrt[3]{\rho} : \sqrt[3]{\rho'}$ represent the x, y, z coordinates of the six points of contact with the oval; these latter therefore, by (1), lie in a conic.

[We have also received a solution of the Question from Professor Cremona, agreeing in principle with that by Professor Cayley.]

1321. (Proposed by the EDITOR.)—If two marbles are thrown at random on the floor of a rectangular room, what is the chance that they will rest at a distance apart less than a given distance?

Solution by M. W. CROFTON, B.A.

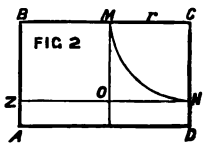


Let a be the length (AD) of the room ABCD, b its breadth (AB), and r the given distance. We shall get all the possible cases by making one point move over the whole surface of the rectangle (Fig. 1), and, in each of its positions (P), taking the other point at random in the rectangle GBCF; and it is easy to see that this is equivalent to taking the other point at random in the rectangle PECF, and doubling the result. To find, then, the number of favourable cases, we shall take P in any position, and draw a circle with P as centre and r as radius; then the arc VW of this circle, which falls inside the rectangle PECF, separates the favourable positions of the second point from the unfavourable; and if dS is the element of the surface at P, the measure (F) of the favourable cases will be

$$F = 2\iint (dS\,.\,EVWP).$$

We shall find it more convenient, however, to find the differential of F with regard to r; if r receive an increment dr, the corresponding increment of F, for the particular position of P, will be $2dS\,.\,\text{arc VW}\,.\,dr$,

therefore $$\frac{dF}{dr} = 2\iint \text{arc VW}\,.\,dS.$$



We have then only to determine the value of the arc VW in terms of the coordinates of P for each different position of P.

1. *In the first case, when r is less than a or b.* Form (Fig. 2) a square CO whose side is r, it will be easily seen that

if P is in AO, arc VW $= r\dfrac{\pi}{2}$,

if P is in BO, arc VW $= r\sin^{-1}\dfrac{y}{r}$,

if P is in DO, arc VW $= r\sin^{-1}\dfrac{x}{r}$

if P is in the quadrant CMN, arc VW $= 0$,

if P is in the mixtilinear space MON, arc VW $= r\left(\sin^{-1}\dfrac{y}{r} - \cos^{-1}\dfrac{x}{r}\right)$.

N.B.—In each of the 4 rectangles in the figure the axes of reference taken are its two sides which belong to the original rectangle; *e.g.*, when P is in BO, the axes are BM, BZ.

$$\text{Hence } \frac{1}{2}\frac{dF}{dr} = r\int_0^{a-r}\int_0^{b-r}\frac{\pi}{2}\,dx\,dy + r\int_0^{a-r}dx\int_0^r dy\,\sin^{-1}\frac{y}{r}$$

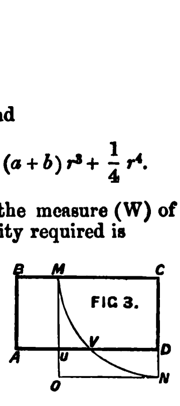
$$+ r\int_0^{b-r}dy\int_0^r dx\,\sin^{-1}\frac{x}{r} + r\int_0^r dx\int_{(r^2-x^2)^{\frac{1}{2}}}^{r} dy\left(\sin^{-1}\frac{y}{r} - \cos^{-1}\frac{x}{r}\right)$$

Performing the integrations, which are very easy, we find

$$\frac{1}{2}\frac{dF}{dr} = \frac{\pi}{2}abr - (a+b)\,r^2 + \frac{1}{2}r^3, \;\therefore\; F = \frac{\pi}{2}\,abr^2 - \frac{2}{3}(a+b)\,r^3 + \frac{1}{4}\,r^4.$$

No constant is added, as F should vanish with r. Now the measure (W) of the whole number of cases is $\frac{1}{4}\,a^2b^2$. Hence the probability required is

$$p = \frac{F}{W} = \frac{r^2}{a^2b^2}\left\{\pi ab - \frac{4}{3}(a+b)\,r + \frac{1}{2}\,r^2\right\}.$$



2. *When r is less than a and greater than b.* Here we have to integrate (Fig. 3) over the spaces BU and MUV; then

$$\frac{1}{2}\frac{dF}{dr} = \int_0^{a-r} dx \int_0^b dy\,\sin^{-1}\frac{y}{r} + r\int_0^b dy \int_{(r^2-y^2)^{\frac{1}{2}}}^{r} dx\left(\sin^{-1}\frac{y}{r} - \cos^{-1}\frac{x}{r}\right)$$

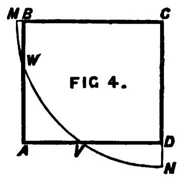
$$= ar\left\{b\,\sin^{-1}\frac{b}{r} + (r^2-b^2)^{\frac{1}{2}} - r\right\} - \frac{b^2}{2}\,r;$$

$$\therefore\; \frac{1}{2}\,F = ab\int \sin^{-1}\frac{b}{r}\,rdr + a\int r\,(r^2-b^2)^{\frac{1}{2}}\,dr - \frac{ar^3}{3} - \frac{b^2r^2}{4},$$

$$\therefore\; F = abr^2\,\sin^{-1}\frac{b}{r} + \frac{a}{3}(2r^2+b^2)\,(r^2-b^2)^{\frac{1}{2}} - \frac{2ar^3}{3} - \frac{b^2r^2}{2} + C.$$

The constant C is found by considering that if $r = b$, the value of F is the same as in case 1, hence we find that $C = \frac{1}{12}\,b^4$;

$$\therefore\; p = \frac{2r^2}{ab}\sin^{-1}\frac{b}{r} + \frac{2}{3}\cdot\frac{2r^2+b^2}{ab^2}\,(r^2 - b^2)^{\frac{1}{2}} - \frac{4}{3}\cdot\frac{r^3}{ab^2} - \frac{r^2}{a^2} + \frac{1}{6}\cdot\frac{b^2}{a^2}.$$



3. *When r is greater than a.* Here we have to integrate (Fig. 4.) over the space VAW; then

$$\frac{1}{2}\cdot\frac{dF}{dr} = r\int_{(r^2-a^2)^{\frac{1}{2}}} dy \int_{(r^2-y^2)^{\frac{1}{2}}}^{a} dx\left(\sin^{-1}\frac{y}{r} - \cos^{-1}\frac{x}{r}\right)$$

$$= r\left\{ab\left(\sin^{-1}\frac{b}{r} - \cos^{-1}\frac{a}{r}\right) + a\,(r^2-b^2)^{\frac{1}{2}} + b\,(r^2-a^2)^{\frac{1}{2}} - \frac{a^2}{2} - \frac{b^2}{2} - \frac{r^2}{2}\right\}$$

$$\therefore\; \frac{1}{2}\,F = ab\int \sin^{-1}\frac{b}{r}\,rdr + ab\int \sin^{-1}\frac{a}{r}\,rdr - \frac{\pi}{2}\,ab\int rdr$$

$$+ a\int (r^2-b^2)^{\frac{1}{2}}\,rdr + b\int (r^2-a^2)^{\frac{1}{2}}\,rdr - \frac{a^2r^2}{4} - \frac{b^2r^2}{4} - \frac{r^4}{8};$$

$$\therefore\; F = abr^2\left(\sin^{-1}\frac{a}{r} + \sin^{-1}\frac{b}{r}\right) - \frac{\pi}{2}\,abr^2 + \frac{a}{3}(2r^2+b^2)\,(r^2-b^2)^{\frac{1}{2}}$$

$$+ \frac{b}{3}(2r^2+a^2)\,(r^2-a^2)^{\frac{1}{2}} - \frac{r^2}{2}(a^2+b^2) - \frac{r^4}{4} + C.$$

To determine C, let $r = a$, then the value of F should be the same as in case (2), hence we find that $C = \frac{1}{12}(a^4+b^4)$;

$$\therefore p = \frac{2r^2}{ab}\left(\sin^{-1}\frac{a}{r} + \sin^{-1}\frac{b}{r}\right) - \frac{\pi r^2}{ab} + \frac{2}{3}\cdot\frac{2r^2+a^2}{a^2b}(r^2-a^2)^{\frac{1}{2}}$$
$$+ \frac{2}{3}\cdot\frac{2r^2+b^2}{ab^2}(r^2-b^2)^{\frac{1}{2}} - r^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right) - \frac{r^4}{2a^2b^2} + \frac{1}{6}\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right).$$

NOTE.—If we call the function which expresses the probability in the first case, $F(a, b, r)$; in the second, $f(a, b, r)$; and in the third, $\phi(a, b, r)$; we shall have the following remarkable relation between the forms of these functions,

$$F(a,b,r) + \phi(a,b,r) = f(a,b,r) + f(b,a,r);$$

evidently pointing to some law in the analytical theory of probability, connecting the functions in problems which give rise to several cases, and which would be well worthy of the attention of mathematicians. To take a different class of problem, let us suppose two bags containing m and n balls respectively $(m > n)$: a person draws an arbitrary number of balls (0 being admissible) from each; to find the chance of his drawing an assigned number (x) in all. It will be easily found that if x is between 0 and (n), the chance is $F(m, n, x) = \frac{x+1}{(m+1)(n+1)}$; if x is between (n) and (m), $f(m, n, x) = \frac{1}{m+1}$; if x is between (m) and $(m+n)$, $\phi(m, n, x) = \frac{m+n-x+1}{(m+1)(n+1)}$.

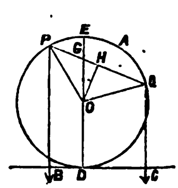
Here also $\quad (m, n, x) + \phi(m, n, x) = f(m, n, x) + f(n, m, x)$,

and there can be no doubt that analogous results will be found to hold in similar cases.

[A solution by Mr. WOOLHOUSE will be found in the *Educational Times* for March, 1863.]

1765. (Proposed by EXHUMATUS.)—Two weights are attached by short strings of given length to the surface of a cylinder; determine the position of equilibrium of the cylinder when resting on a horizontal plane, the weights resting on the surface of the cylinder.

Solution by E. L. ROSOLIS.



Let O be the centre of the cylinder, P and Q the weights, A the point to which the strings AP and AQ are attached; join PQ, OP, and OQ; draw PB, QC, and EOD perpendicular to the horizontal plane BC, and OH perpendicular to PQ. Let θ = inclination of PQ to the horizon = $\angle$ EOH, r = radius of cylinder, $AP = m$, $AQ = n$, $2a$ = angle whose circular measure is $\frac{m+n}{r} = \angle$ POQ. In the position of equilibrium the centre of gravity (G) of P and Q must be in the vertical line DE, therefore

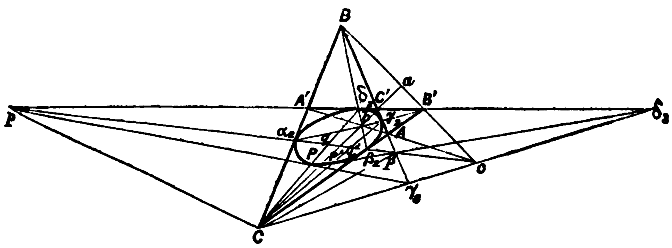
$$\frac{P}{Q} = \frac{QG}{PG} = \frac{CD}{BD} = \frac{\sin(a+\theta)}{\sin(a-\theta)}, \text{ whence } \tan\theta = \frac{P-Q}{P+Q}\tan a.$$

1789. (Proposed by Professor WHITWORTH.) — Let abc be the triangle formed by the three diagonals AA′, BB′, CC′ of a quadrilateral; and let Aa, $B'b$, $C'c$ meet A′BC in $\alpha_1, \alpha_2, \alpha_3$; $A'a$, Bb, $C'c$ meet AB′C in $\beta_1, \beta_2, \beta_3$; $A'a$, $B'b$, Cc, meet ABC′ in $\gamma_1, \gamma_2, \gamma_3$; Aa, Bb, Cc, meet A′B′C′ in $\delta_1, \delta_2, \delta_3$; then $\alpha_2, \beta_2, \gamma_2, \delta_2$, are the points of contact of the "*Critical* Inscribed Conic" of the quadrilateral CA′C′A (this may be taken as the *Definition* of the "*Critical* Inscribed Conic"), and Professor Cayley has given in Quest. 1751 (*Reprint*, Vol. IV., p. 38), a construction for finding four other points on the conic and the tangents thereat.

It is required to show that, by the ruler alone, *eight* other points on this conic, and the tangents thereat, may be found by the following

Construction.—Let $\alpha_2\beta_2$ meet C′A′ in p; join $p\gamma_3$ cutting $C\delta_2$ in P; then P is a point on the curve, and Pp the tangent at P. Similarly, *seven* other points may be found, and the tangents constructed thereat.

Solution by J. DALE.



From the quadrilateral CA′ B′c Bδ_3 it is evident that $Cc\gamma_3\delta_3$ is a harmonic range; therefore, as $\alpha_2\beta_2$ passes through c, CqPδ_2 is also a harmonic range; and as C is the polar of $\alpha_2\beta_2$, and δ_2 one of the points in which Cq meets the curve, it follows that P is the other point; and as p is the pole of $C\delta_2$, and $p\delta_2$ *one* of the tangents from p, therefore pP is the *other*. Again, let $\alpha_2\beta_2$ produced meet C′A in p'; join $p'\delta_3$ cutting $C\gamma_2$ in P′; then p'P′ touches the conic in P′. The proof is the same as before, since the line $C\gamma_2$ is harmonically divided in q', P′. So also may other six points be found on the lines $C'\alpha_2$, $C'\beta_2$; $A\alpha_2$, $A\delta_2$; $A'\beta_2$, $A'\gamma_2$; respectively.

II. *Solution by the* PROPOSER.

Take the given quadrilateral as quadrilateral of reference for that system of quadrilinear coordinates where the coordinates of every point are subject to the identity
$$\alpha+\beta+\gamma+\delta \equiv 0 \quad \ldots\ldots\ldots\ldots (i).$$
Then (as I have shown in my paper on Quadrilinear Coordinates, *Messenger of Mathematics*, Vol. III., p. 174, Art. 32.) the critical conic inscribed in the quadrilateral CAC′A′ is represented by either of the equivalent equations
$$(\alpha-\gamma)^2 = 4\beta\delta, \text{ or } (\beta-\delta)^2 = 4\alpha\gamma \quad \ldots\ldots\ldots\ldots (ii),$$
or by any equation formed by combining either of these with the identity (i). Now since $\alpha-\gamma \equiv 2\alpha+\beta+\delta$, the first equation to the conic may be written
$$(2\alpha+\beta+\delta)^2 = 4\beta\delta, \text{ or } (2\alpha+\beta)^2+\delta\,(4\alpha-2\beta+\delta) = 0 \ldots\ldots (iii),$$
which shows that ($\delta = 0$) and ($4\alpha-2\beta+\delta = 0$) represent tangents at the

extremities of the chord $(2\alpha+\beta=0)$, or $C\delta_2$. Now p is given by $2\alpha-\beta+\delta=0$ (or $\alpha_2\beta_2$) and $\delta=0$ (or $C'A'$), and therefore lies on the line $(4\alpha-2\beta+\delta=0)$. And this equation is also satisfied when $\alpha-\beta=0$ and $\gamma=0$, i.e. at γ_3. Hence $4\alpha-2\beta+\delta=0$ represents the line $p\gamma_3$, which therefore touches the conic at its point of intersection with $C\delta_2$.

On Triangular Symmetry. By W. K. Clifford.

We make the properties of a conic intuitive by studying it under the form of a circle; or those of a quadrilateral, by studying it under the form of a square. This simplification depends upon the projective property of a right angle, viz., that it divides harmonically the chord at infinity of a circle. By means of this property we interpret as general, propositions whose truth we see intuitively through the symmetry of the figure. This kind of symmetry (that of a circle or square) I call the ***symmetry of the right angle***, or ***rectangular symmetry***.

From the symmetry of two lines we ascend immediately to the symmetry of three lines, or of the equilateral triangle. This is exemplified in the Rhombohedral System of Crystals, just as Rectangular Symmetry is exemplified in the Pyramidal System. I want in this note to call attention to the uses of Triangular Symmetry in presenting general propositions under an intuitive form.

The projective property of an equilateral triangle is this: ***it determines on the line at infinity a point-cubic whose Hessian is the circular points.*** Given four lines, we may project one of them to infinity so that the other three shall form an equilateral triangle; for we have only to construct the Hessian of the point-cubic determined upon that one by the other three, and then project this Hessian into the circular points. Similar problems are

B. To project at once a conic and triangle into a circle and an equilateral triangle.

C. To project at once two conics and a triangle into two rectangular hyperbolæ and an equilateral triangle.

D. To project at once two triangles into equilateral triangles.

In each of these we have the problem of finding the line which has to be projected to infinity; this problem admits, in the three cases respectively, of 43, 76, and 108 solutions. The *triangle* might of course be replaced by a *cubic*, to be so projected that its asymptotes should form an equilateral triangle; but this case is not particularly interesting. Every cubic may be projected into a perfectly symmetrical form in this way;—the three real inflexions of the cubic lie on a certain straight line, and determine a point-cubic upon it; let the Hessian of this point-cubic be projected into the circular points at infinity. Then the cubic is symmetrically situated in respect of an equilateral asymptotic triangle.

The form in Fig. 1 may be called *inscribed*, that in Fig. 2 *escribed*. The inscribed cubic may have no oval nor double point, and then will lie entirely without the triangle.

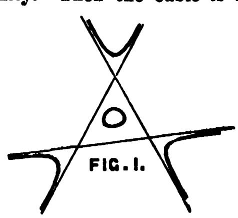


FIG. 1.

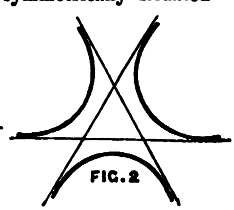


FIG. 2

Cubics with a crunode cannot be thus symmetrized; but their reciprocals can. In fact, every three-cusped quartic can be projected into a hypocycloid of three branches. For, any four points can be projected into any other four points; if then the three cusps and the intersection of the cuspidal tangents be projected respectively into the vertices and centre of an equilateral triangle, the thing is done. This proof is Professor CAYLEY'S. We learn in this way that, in any three-cusped quartic, the cuspidal tangents and the lines joining the cusps determine on the double tangent two point-cubics, whose common Hessian is the points of contact of that tangent.

Trinodal quartics have four double tangents, which are all real as lines when the nodes are real, but one is ideal (I borrow this convenient expression from Poncelet) or has imaginary contacts. If the three pairs of nodal tangents divide the ideal double tangent in the same anharmonic ratio, the quartic can be projected into a hypotrochoid, the contacts of the ideal tangent being then the circular points at infinity. The curve has many remarkable properties, which can be recognised at once from the symmetry of the projected figure.

Higher orders of symmetry are special cases. Thus quartic symmetry (as of a regular octagon) requires that the point-quartic shall be an harmonic system, so that its cubinvariant vanishes. Quintic symmetry (regular pentagon) requires a point quintic of the form ax^5+fy^5. The conditions that the quintic may be reduced to this form are that the invariants K of the eighth degree and L of the twelfth shall separately vanish (see Professor SYLVESTER'S admirable *Trilogy*, *Phil. Trans.* 1864, p. 619).

The cube and sphere are examples of the symmetry of the *cubangle*, whose projective property is that it determines on the plane at infinity a conjugate triad of the imaginary circle. One is naturally led to seek for the projective property of a regular tetrahedron. It determines on the plane at infinity four straight lines, x, y, z, w; and if we assume the identical relation

$x+y+z+w\equiv 0$, then the equation of the imaginary circle is

$x^2+y^2+z^2+w^2=0$, as is easily shown by the consideration that each face is an equilateral triangle. Many interesting properties of this conic will be found proved in the solution to Question 1690.

1741. (Proposed by J. GRIFFITHS, M.A.)—Let P be the point of intersection of the perpendiculars, and G the centre of gravity, of any triangle ABC; prove that the nine-point circles of all the triangles that can be formed from the points A, B, C, P, G, together with the maximum ellipse that can be inscribed in the triangle ABC, all pass through a common point, viz., the centre of the equilateral hyperbola passing through A, B, C, P, G.

Solution by J. DALE; E. MCCORMICK; *and others.*

The equilateral hyperbola passing through A, B, C, P, G, circumscribes each one of the ten triangles ABC, BPC, CPA, APB, BGC, CGA, AGB, PGA, PGB, PGC; and as the centre of an equilateral hyperbola circumscribing a triangle lies upon the nine-point circle of the triangle, it follows that the nine-point circles of the ten triangles pass through one point, viz., the centre of the equilateral hyperbola ABCPG.

It now remains to be shown that the maximum inscribed ellipse also passes through this point. The equation of the hyperbola is

$$\frac{b^2-c^2}{a\alpha}+\frac{c^2-a^2}{b\beta}+\frac{a^2-b^2}{c\gamma}=0 \quad\text{.....................} (1),$$

and the coordinates of the centre are connected by the equations

$$\frac{a\alpha}{(b^2-c^2)^2}=\frac{b\beta}{(c^2-a^2)^2}=\frac{c\gamma}{(a^2-b^2)^2} \quad\text{..................} (2).$$

The equation of the maximum ellipse is

$$\sqrt{(a\alpha)}+\sqrt{(b\beta)}+\sqrt{(c\gamma)}=0 \quad\text{.....................} (3).$$

Now it is evident that the values of α, β, γ, obtained from (2), satisfy (3), hence the centre of the equilateral hyperbola lies on the ellipse; therefore the centre of the hyperbola is a point common to each one of the ten nine-point circles, and to the maximum inscribed ellipse.

1675. (Proposed by W. K. CLIFFORD.)—If a triangle abc be the reciprocal of ABC in respect of a parabola whose parameter is $4m$; and if n_1, n_1, n_3 be the normals at the vertices of diameters through ABC; then

$$\frac{(\text{area of } abc)^2}{bc\,.\,ca\,.\,ab}=\frac{2m^2}{n_1n_2n_3}(\text{area of ABC}).$$

Solution by J. DALE; E. FITZGERALD; *and others.*

The equation of the parabola being $y^2=4mx$, let the coordinates o A, B, C be (h_1, k_1), (h_2, k_2), (h_3, k_3), respectively; then we have

$$2\,(\text{area of ABC})=\begin{vmatrix} h_1 & k_1 & 1\\ h_2 & k_2 & 1\\ h_3 & k_3 & 1\end{vmatrix} \quad\text{........................} (1);$$

also $n_1^2=k_1^2+4m^2$, $n_2^2=k_2^2+4m^2$, $n_3^2=k_3^2+4m^2$; and the equations to bc, ca, ab (the polars of A, B, C) are, respectively,

$$k_1y=2m\,(x+h_1),\quad k_2y=2m\,(x+h_2),\quad k_3y=2m\,(x+h_3).$$

From these equations the coordinates of b, c are found to be

$$\frac{\begin{vmatrix} h_3 & k_3\\ h_1 & k_1\end{vmatrix}}{\begin{vmatrix} k_3 & 1\\ k_1 & 1\end{vmatrix}},\quad 2m\frac{\begin{vmatrix} h_3 & 1\\ h_1 & 1\end{vmatrix}}{\begin{vmatrix} k_3 & 1\\ k_1 & 1\end{vmatrix}};\quad \frac{\begin{vmatrix} h_1 & k_1\\ h_2 & k_2\end{vmatrix}}{\begin{vmatrix} k_1 & 1\\ k_2 & 1\end{vmatrix}},\quad 2m\frac{\begin{vmatrix} h_1 & 1\\ h_2 & 1\end{vmatrix}}{\begin{vmatrix} h_1 & 1\\ k_2 & 1\end{vmatrix}};$$

and from these values of the coordinates, we find

$$(bc)^2=(k_1^2+4m^2)\begin{vmatrix} h_1 & k_1 & 1\\ h_2 & k_2 & 1\\ h_3 & k_3 & 1\end{vmatrix}^2 \div \left(\begin{vmatrix} k_3 & 1\\ k_1 & 1\end{vmatrix}\,\begin{vmatrix} k_1 & 1\\ k_2 & 1\end{vmatrix}\right);$$

vith similar values for $(ca)^2$, $(ab)^2$; hence we obtain

$$\frac{bc}{n_1}\cdot\frac{ca}{n_2}\cdot\frac{ab}{n_3}=\frac{\begin{vmatrix} h_1 & k_1 & 1 \\ h_2 & k_2 & 1 \\ h_3 & k_3 & 1 \end{vmatrix}^2}{\left(\begin{vmatrix} k_2 & 1 \\ k_3 & 1 \end{vmatrix}\begin{vmatrix} k_3 & 1 \\ k_1 & 1 \end{vmatrix}\begin{vmatrix} k_1 & 1 \\ k_2 & 1 \end{vmatrix}\right)^2} \quad\ldots\ldots\ldots (2.)$$

Moreover the area of the triangle abc is

$$m\frac{\begin{vmatrix} \begin{vmatrix} h_2 & k_2 \\ h_3 & k_3 \end{vmatrix} & \begin{vmatrix} h_2 & 1 \\ h_3 & 1 \end{vmatrix} & \begin{vmatrix} k_2 & 1 \\ k_3 & 1 \end{vmatrix} \\ \begin{vmatrix} h_3 & k_3 \\ h_1 & k_1 \end{vmatrix} & \begin{vmatrix} h_3 & 1 \\ h_1 & 1 \end{vmatrix} & \begin{vmatrix} k_3 & 1 \\ k_1 & 1 \end{vmatrix} \\ \begin{vmatrix} h_1 & k_1 \\ h_2 & k_2 \end{vmatrix} & \begin{vmatrix} k_1 & 1 \\ k_2 & 1 \end{vmatrix} & \begin{vmatrix} k_1 & 1 \\ k_2 & 1 \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} k_2 & 1 \\ k_3 & 1 \end{vmatrix}\begin{vmatrix} k_3 & 1 \\ k_1 & 1 \end{vmatrix}\begin{vmatrix} k_1 & 1 \\ k_2 & 1 \end{vmatrix}} = \frac{m\begin{vmatrix} h_1 & k_1 & 1 \\ h_2 & k_2 & 1 \\ h_3 & k_3 & 1 \end{vmatrix}^2}{\begin{vmatrix} k_2 & 1 \\ k_3 & 1 \end{vmatrix}\begin{vmatrix} k_3 & 1 \\ k_1 & 1 \end{vmatrix}\begin{vmatrix} k_1 & 1 \\ k_2 & 1 \end{vmatrix}} \quad (3).$$

The theorem in question follows at once from (1), (2), (3).

1672. (Proposed by J. TAYLOR.)—Give simple geometrical constructions for (1) inscribing the greatest triangle in an ellipse, and (2) circumscribing the least triangle about an ellipse.

Solution by the Rev. J. L. KITCHIN, M.A.

1. The greatest triangle in a circle is an equilateral triangle, and a side bisects the radius perpendicular to it. If the circle and triangle be projected orthogonally on a plane through the diameter parallel to a side, the circle becomes an ellipse, and the triangle is the greatest triangle in the ellipse. Its side also passes through the *middle* point of the semi-minor axis. If therefore we draw, through the middle point of the semi-minor axis, a line parallel to the major axis, and join the points in which it meets the curve with the further end of the minor axis, the triangle so formed is the *greatest* triangle which can be inscribed in the ellipse.

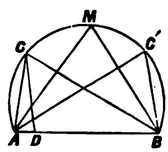
2. The *least* triangle about a circle is an equilateral triangle. If a circle be projected orthogonally on a plane through a diameter, the ellipse so formed bears the same relation to the circle, whence it is derived, as subsists between an ellipse and the circle on its major axis. Hence we may call the ellipse the *projection* of the circle, remembering that, when the projection is made, the plane is turned about the diameter until it coincides with the plane of the circle. Now, projecting the circle and its circumscribing triangle on a plane passing through a diameter parallel to one of the sides of the triangle, we obtain our ellipse and its least circumscribing triangle. The side parallel to the diameter is not changed in the projection. Hence the following construction:—Describe the circle on the major axis of the given ellipse, and about it an equilateral triangle. Through either end of the minor axis

draw a tangent to the ellipse, bisected in that point, and equal to the side of the triangle. From the ends of this line draw tangents to the ellipse, and produce them to meet; the triangle so formed is the least triangle circumscribing the ellipse.

3. The Solution of Quest. 1781 may also be effected very simply by making use of orthogonal projection. For all that is necessary is to solve the problem in the case of the circle on the major axis of the ellipse. Then drawing ordinates from the angular points on the circle, we obtain, by their intersection with the ellipse, the angular points sought of the greatest triangle in the ellipse; for the latter is the *projection* of the former.

1807. (Proposed be R. TUCKER, M.A.)—From the ends of a diameter of a semicircle, two persons walk in random directions within the semicircle; find the chance of their paths intersecting inside the circle.

Solution by J. DALE; E. FITZGERALD; *and others.*



Suppose one walks along any line AC making an angle θ with AB, the path of the other, starting from B, will intersect AC so long as the angle ABC is less than $\frac{1}{2}\pi-\theta$; the chance of this is $(\frac{1}{2}\pi-\theta)\div\frac{1}{2}\pi$, and the mean value of this chance, supposing θ to vary from 0 to $\frac{1}{2}\pi$ is

$$\Sigma\left(\frac{\frac{1}{2}\pi-\theta}{\frac{1}{2}\pi}\cdot\frac{\Delta\theta}{\frac{1}{2}\pi}\right),\ \text{or}\ \frac{4}{\pi^2}\int_0^{\frac{1}{2}\pi}(\tfrac{1}{2}\pi-\theta)\,d\theta=\tfrac{1}{2};$$

therefore, the chances of intersecting inside or outside the circle are equal.

[*Otherwise:* Let AMB be *any circular segment* (we have adapted Mr. Dale's figure to this form of the Problem), M the middle point of the arc, AC and AC′ any two lines equally inclined to AM; then, supposing all directions within the segment to be equally probable, and observing that the number of positions of AC′ is equal to those of AC, the measure (F) of the whole number of favourable cases will be $F=\frac{1}{2}(\text{arc AC}+\text{arc AC}')\times\text{arc AB}$ $=\frac{1}{2}(\text{arc AB})^2$, and the measure W of the whole number of cases is $W=(\text{arc AB})^2$; hence the required probability is $p=\dfrac{F}{W}=\frac{1}{2}$.

Similar Problems may be proposed involving other curves. For instance, if AMB were an *ellipse*, AB ($=2a$) the major axis, $2b$ the minor axis, and CD perpendicular to AB, we should have $a^2:b^2=\text{AD}\,.\,\text{DB}:\text{CD}^2$ $=\cot\text{CAD}$ (or $\cot\theta$) : $\tan\text{CBD}$; therefore $\text{CBD}=\tan^{-1}\left(\dfrac{b^2}{a^2}\cot\theta\right)$.

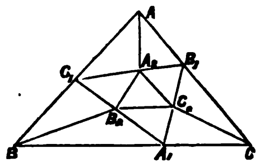
Hence the required probability will be expressed by the integral

$$\frac{4}{\pi^2}\int_0^{\frac{1}{2}\pi}\tan^{-1}\left(\frac{b^2}{a^2}\cot\theta\right)d\theta,$$

which, when $a=b$, becomes the integral given in Mr. Dale's Solution.]

1726. (Proposed by W. GODWARD.)—Let O_1, O_2, O_3 be the centres of the escribed circles having internal contact with the sides BC, CA, AB respectively of the triangle ABC, and P any other point in the same plane; then (1) if R_1, R_2, R_3 be the intersections of AP, BP, CP with BC, CA, AB, respectively, the triad (O_1R_1, O_2R_2, O_3R_3) will be concurrent; and (2) if S_1, S_2, S_3 be the intersections of O_1P, O_2P, O_3P with BC, CA, AB, respectively, the triad (AS_1, BS_2, CS_3) will be concurrent.

Solution by S. BILLS; W. H. LAVERTY; J. DALE; *and the* PROPOSER.



The first part of the question will, evidently, only be a particular case of the following much more general and comprehensive theorem. Let ABC be any plane triangle, and from its angular points let lines be drawn through *any* assumed point, meeting the opposite sides in A_1, B_1, C_1; describe the triangle $A_1B_1C_1$, and from its angular points draw lines through *any other* assumed point, meeting the opposite sides in A_2, B_2, C_2; then will the lines AA_2, BB_2, CC_2 be concurrent. For, from the triangles AA_2B_1, AA_2C_1; &c., &c.; we readily obtain

$$\frac{\sin A_2AC_1}{\sin A_2AB_1} = \frac{A_2C_1 \cdot AB_1}{A_2B_1 \cdot AC_1}, \quad \frac{\sin B_2BA_1}{\sin B_2BC_1} = \frac{B_2A_1 \cdot BC_1}{B_2C_1 \cdot BA_1},$$

$$\frac{\sin C_2CB_1}{\sin C_2CA_1} = \frac{C_2B_1 \cdot CA_1}{C_2A_1 \cdot CB_1};$$

hence, multiplying together these three equations, and remembering that $A_2C_1 \cdot B_2A_1 \cdot C_2B_1 = A_2B_1 \cdot B_2C_1 \cdot C_2A_1$, $AB_1 \cdot BC_1 \cdot CA_1 = AC_1 \cdot BA_1 \cdot CB_1$, the criterion of the concurrence of AA_1, BB_1, CC_1 is satisfied, *viz.*,

$$\sin A_2AC_1 \cdot \sin B_2BA_1 \cdot \sin C_2CB_1 = \sin A_2AB_1 \cdot \sin B_2BC_1 \cdot \sin C_2CA_1.$$

[Mr. BILLS gives at length an independent proof of the property (2), with another diagram. But from what is given above, we may, in fact, prove the following theorem, which includes, as particular cases, *both* those in the Question, together with that in Question 1616 (*Reprint*, Vol. III., p. 29).

If of three triangles, each inscribed in the preceding, any two are in homology with the third, they are in homology with each other.

For let the figure represent any three such triangles; then, putting

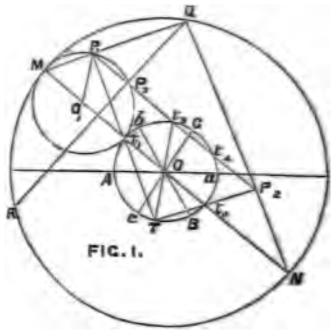
$$\frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = p, \quad \frac{A_2B_1}{A_2C_1} \cdot \frac{B_2C_1}{B_2A_1} \cdot \frac{C_2A_1}{C_2B_1} = q,$$

$$\frac{\sin A_2AC_1}{\sin A_2AB_1} \cdot \frac{\sin B_2BA_1}{\sin B_2BC_1} \cdot \frac{\sin C_2CB_1}{\sin C_2CA_1} = r,$$

and introducing appropriate *signs*, Mr. BILLS' equations give $pqr = -1$. If therefore any two of the three quantities (p, q, r) are equal to -1, so likewise is the third; that is to say, (by Ceva's theorem) if any two of the three triangles (whose criterions of homology, in pairs, are $p = -1$, $q = -1$, $r = -1$) are homologous (or co-polar) with the third, they are homologous with each other.

After the above was in type we observed that the theorem is proved, but in a different manner (by the method of anharmonic ratio), in Townsend's *Modern Geometry*, Vol. II., p. 144.]

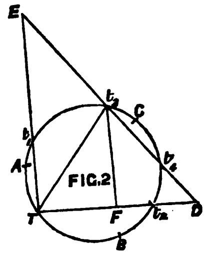
SOLUTIONS OF QUESTIONS 1713, 1716, 1734; AND DEMONSTRATIONS OF SOME OF THE PROPERTIES OF THE REGULAR TRICUSP ENUNCIATED IN STEINER'S PAPER (*Reprint* Vol., III., pp. 97—100). BY J. DALE.



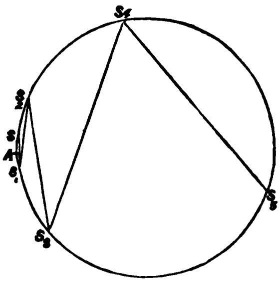
1. Let O (Fig. 1) be the common centre of the circles ABC, MQN, the radius of the latter being three times that of the former; draw any radius Ot_1M making an angle θ with the radius OA; take $OO_1 = \frac{2}{3}OM$; from O_1 draw $O_1P_1 = OM$ and making an angle 3θ with OO_1: then it is evident that P_1 is a point on the hypocycloid of three branches (or *tricusp*) touching the interior circle in A, B, C, the points of trisection of the circumference, and having A*a*, B*b*, C*c* for cuspidal tangents. Join P_1M, P_1t_1, and produce P_1t_1 to cut ABC in T; then from the nature of roulettes, P_1M and P_1t_1T are respectively the normal and tangent to the tricusp at P_1. Produce MO, cutting the circles in t_2, N, and join TO; then the triangles $P_1O_1t_1$, TOt_1, are equal in all respects; hence the arc $At_1 = \frac{1}{3}AT$, and the chord $P_1t_1 = Tt_1$; and since the arc AB = BC = CA, we have $Ct_1 = \frac{1}{3}CT$, and $BCt_1 = \frac{1}{3}BACT$. Therefore (I) *the arcs into which the inscribed circle is divided by the tangent are divided in the ratio of* 1 : 2 *by the cuspidal tangents; and* (II) *the chord intercepted by the inscribed circle on any tangent is bisected at the point nearest the point of contact of the tricusp and circle; and conversely, any line satisfying these conditions is a tangent to the tricusp.*

2. Join Tt_2, then the arc $AB = 2aB$, and $AT = 2At_1 = 2at_2$; hence $Bt_2 = \frac{1}{3}BT$, and therefore, by (I), Tt_2 is a tangent to the tricusp; and making $t_2P_2 = t_2T$, P_2 is, by (II), the point of contact. Join P_1P_2, cutting the inscribed circle in t_3, t_4; then as P_1P_2 is parallel to t_1t_2 and $= 2t_1t_2$, t_3 lies on TO, and the arc $t_1t_3 = t_2t_4 = Tt_2 = \frac{1}{3}Tt_4$; therefore $At_3 = \frac{1}{3}At_4$, also $At_3C = \frac{1}{3}At_4C$, therefore $Ct_3 = \frac{1}{3}Ct_4$; hence P_1P_2 is a tangent, and if $t_3P_3 = t_3t_4$, P_3 is the point of contact. It may also be readily shown from the figure that the normals at P_1, P_2, P_3 meet in a point Q on the circumscribed circle. Therefore, (III) *tangents at right angles intersect on the inscribed circle;* (IV) *the chord of contact* P_1P_2 *is of constant length* (4*r*), *and* (V) *is also a tangent to the curve;* (VI) *the normals at any three points, the tangents at which form a right-angled triangle, meet in the circumscribed circle;* (VII) *the chord intercepted by the circumscribed circle on any normal is divided in the ratio of* 1 : 2 *at the point of contact.*

3. QM is parallel to Tt_2, and if the cuspidal tangent B*b* be produced to meet the circumscribed circle, it will divide the arc QM in the ratio of 1 : 2; therefore (VIII) *the normal to the tricusp at any point is a tangent to another tricusp touching the circumscribed circle at the cusps of the former; and conversely, the tangent to the tricusp at any point is a normal to another tricusp having the inscribed circle of the former for its circumscribed circle, the cusps of the one corresponding to the points of contact of the other.* In other words, *the evolute and involute of the tricusp are similar tricusps.*



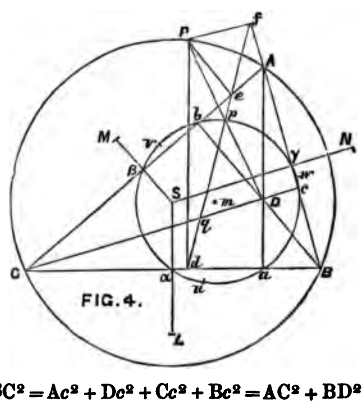
4. Let ABC (Fig. 2) be the inscribed circle of a tricusp; A, B, C being the points of contact; Tt_1, Tt_2 two tangents at right angles; through any point t_3 on the circle draw a straight line so that the portion DE intercepted between Tt_1, Tt_2 may be bisected in t_3. Join Tt_3; then the arc $t_1t_3 = \frac{1}{3}Tt_2t_4$, and as $At_1 = \frac{1}{3}AT$, we have $At_1t_3 = \frac{1}{3}At_2t_4$; therefore, by (I), DE is a tangent to the tricusp. Draw t_3F parallel to ET, then the angle Tt_3D is bisected by t_3F. Therefore, (IX) *if the intercept of a line between two rectangular tangents is bisected by a point in the inscribed circle, it is enveloped by the tricusp*; and, (X) *if the vertex of a variable angle moves on the inscribed circle, and one of the sides turns round a fixed point on the circle, while the bisectors are constantly parallel to a fixed pair of rectangular tangents intersecting at the point; then the other side of the variable angle is enveloped by the tricusp.*



5. Let ss_1 (Fig. 3) be any arc of a circle, and let $sA = \frac{1}{2}s_1A$, then by (I) the chord ss_1 is a tangent to the circumscribed tricusp; and drawing s_1s_2 perpendicular to the diameter through s, or what is the same thing, making the arc $s_1s_2 = 2ss_1$, then the arc $As_2 = 2As_1$, therefore s_1s_2 is a tangent; and drawing according to the same law the chords s_2s_3, s_3s_4, s_4s_5, &c., these chords are also tangents. To find the distance of any one of these points from s, let $ss_1 = a$; then we have $ss_1 = a$, $ss_2 = -a$; $ss_3 = (1+2)\,a$,

$$ss_4 = -(1+4)\,a;\ ss_5 = (1+2+8)\,a,\ ss_6 = -(1+4+16)\,a;$$
$$ss_7 = (1+2+8+32)\,a,\quad ss_8 = -(1+4+16+64)\,a,$$
$$ss_n\ (n \text{ odd}) = \left\{1+2\,(1+2^2+2^4\ldots.+2^{n-3})\right\}a = \tfrac{1}{3}\,(1+2^n)\,a,$$
$$ss_n\ (n \text{ even}) = -\,(1+2^2+2^4\ldots.+2^{n-2})\,a = \tfrac{1}{3}(1-2^n)\,a;$$

and both these forms are included in $ss_n = \frac{1}{3}\left\{1-(-2)^n\right\}a$.



6. Let u, v, w (Fig. 4) be the points of contact of the tricusp; m the centre of the inscribed circle; draw two pairs of rectangular tangents, having their vertices in b and c respectively, and determining by their intersections the quadrilateral ABCD; then, by (IX), AC and AB are bisected in β and γ, the points where they are cut by the circle (m); therefore (m) is the nine-point circle of the triangle ABC, and consequently passes through the middle point of BC; therefore, by (IX), BC is a tangent to the tricusp; and as AD meets BC on the circle (m) at right angles, AD also is a tangent. Also $AD^2 + BC^2 = Ac^2 + Dc^2 + Cc^2 + Bc^2 = AC^2 + BD^2$

$= Ca^2 + Aa^2 + Ba^2 + Da^2 = AB^2 + CD^2 = 16r^2 (\sin^2 C + \cos^2 C) = 16r^2$. If L, M, N, the centres of the circles drawn round BCD, CDA, ADB, be joined, the triangle LMN, as is well known, will be equal and similar to ABC, the corresponding sides BC, MN; CA, NL; AB, LM will be parallel, and the circle (m) will be the common nine-point circle; therefore the sides MN, NL, LM, and the perpendiculars LS, MS, NS, touch a tricusp, having (m) for its inscribed circle, and its points of contact opposite to u, v, w.

7. Let d, e, f be the feet of perpendiculars drawn from any point P in the circumscribed circle on the sides of the triangle ABC; join DP; then def and DP intersect in a point p on the nine-point circle and $Pp = Dp$ (see Quest. 1649, *Reprint*, Vol. III., p. 58), hence the triangles pfP, pqD are equal in every respect, whence $pf = pq$; therefore, since the straight line fq intercepted between the rectangular tangents, cC, cA is bisected in p by the inscribed circle (m), it follows by (IX) that def is a tangent to the tricusp having the circle (m) for its inscribed circle.

1551. (Proposed by the Rev. J. BLISSARD.)—
Using the notation of Spence's Transcendents, viz.,

$$L^n(1+x) = \frac{x}{1^n} - \frac{x^2}{2^n} + \frac{x^3}{3^n} - \frac{x^4}{4^n} + \text{\&c., prove that}$$

$$(1)\ldots\, L^2(1+x) + L^2\left(\frac{1}{1+x}\right) = \tfrac{1}{2}\log^2(1+x);$$

$$(2)\ldots\, L^2\left(\frac{1}{1+x}\right) + L^2\left(\frac{x}{1+x}\right) = \log^2(1+x) - \log x \log(1+x) - \tfrac{1}{6}\pi^2.$$

Solution by the PROPOSER.

It has been shown on p. 59 of Vol. II. of the *Reprint*, that, if B is the Representative of Bernoulli's numbers, we shall have

$$(1+x)^{B} = x^{-1}\log(1+x) = 1 - \tfrac{1}{2}x + \tfrac{1}{3}x^2 - \text{\&c.}\ ; \text{ hence, integrating,}$$

$$\frac{(1+x)^{B+1} - 1}{B+1} = \frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \text{\&c.} = L^2(1+x);$$

also putting $\dfrac{-x}{1+x}$ for x, we have $\dfrac{(1+x)^{-(B+1)} - 1}{B+1} = L^2\left(\dfrac{1}{1+x}\right)$;

therefore $L^2(1+x) + L^2\left(\dfrac{1}{1+x}\right) = 2\left\{(B+1)\dfrac{\log^2(1+x)}{1\,.\,2} + \text{\&c.}\right\}$.

But $(B+1)^n = 0$, (n odd and >1); also $B+1 = B_1 + 1 = \frac{1}{2}$; hence we obtain the formula (1).

Again, since $\frac{1}{x} = \frac{1}{1+x} \cdot \frac{1}{1-(1+x)^{-1}} = \frac{1}{1+x} + \frac{1}{(1+x)^2} + \&c.,$

$$L^2(1+x) = \int \frac{1}{x} \log(1+x)\, dx = \int \log(1+x)\, dx \left\{\frac{1}{1+x} + \frac{1}{(1+x)^2} + \&c.\right\}$$

$$= \tfrac{1}{2}\log^2(1+x) - \left\{\frac{1}{1^2}\cdot\frac{1}{1+x} + \frac{1}{2^2}\cdot\frac{1}{(1+x)^2} + \&c.\right\}$$

$$-\log(1+x)\left\{\frac{1}{1+x} + \frac{1}{2}\cdot\frac{1}{(1+x)^2} + \&c.\right\}$$

$$= -\tfrac{1}{2}\log^2(1+x) + L^2\left(\frac{x}{1+x}\right) + \log(1+x)\log x + C.$$

To determine C, let $x = 1$; then $L^2(2) = -\frac{1}{2}\log^2(2) + L^2(\frac{1}{2}) + C$; but from (1), putting $x = 1$, $L^2(2) + L^2(\frac{1}{2}) = \frac{1}{2}\log^2(2)$;

$$\text{therefore } C = 2L^2(2) = 2\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \&c.\right) = \frac{\pi^2}{6}; \text{ hence}$$

$$L^2(1+x) - L^2\left(\frac{x}{1+x}\right) = \frac{\pi^2}{6} + \log x \log(1+x) - \tfrac{1}{2}\log^2(1+x) \ldots (1').$$

Taking (1′) from (1), we obtain the formula (2).

COROLLARY.—It is easy to show that

$$\log^2(1+x) = 2\left(\Sigma_1 \cdot \frac{x^2}{2} - \Sigma_2 \cdot \frac{x^3}{3} + \Sigma_3 \cdot \frac{x^4}{4} - \&c.\right),$$

where $\Sigma_n = \frac{1}{1} + \frac{1}{2} + \ldots \frac{1}{n}$; hence from (1), by equating coefficients, we obtain the following known property of numbers; viz.,

$$\Sigma_n = \frac{n}{1} - \frac{n(n-1)}{1.2^2} + \frac{n(n-1)(n-2)}{1.2.3^2} - \&c.$$

1784. (Proposed by T. T. WILKINSON, F.R.A.S.)—If from any given point in an ellipse there be drawn an ordinate to either axis, and if from the point where this meets the curve again a diameter be drawn, and to this diameter another ordinate be applied from the given point; then the point in which this ordinate meets the ellipse shall be in the circumference of the circle of curvature at the given point.

Solution by R. WARREN, B.A.; M. COLLINS, B.A.; *and others.*

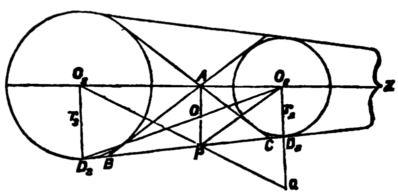
Let C be the centre of the conic, and Q the point in which the circle of curvature at P meets it again. It is a well known theorem, that if a circle intersect a conic, its chords of intersection make equal angles with the axis. In the case of the circle of curvature, the tangent at P is one chord of inter-

section and the line PQ the other; we have, therefore, only to draw PQ, making the same angle with the axis at the tangent at P, and we have the point Q. Now since the tangent at P is an ordinate to the diameter CP, the line PQ is by symmetry an ordinate to the reflexion of CP (with respect to the axis), which is nothing more than the diameter used in the construction given in the Question.

1697. (Proposed by MATTHEW COLLINS, B.A.)—If two circles touch a straight line a, and have a different kind of contact with two other straight lines b and c, prove that the middle point of the distance of a from the point of intersection of b and c lies in a straight line with the centre of one of the two circles and the point where the other circle touches a.

N.B.—The contact of two circles with two straight lines is said to be like, or of the same kind, when the two circles are on the *same side* of *both* lines, or else on *different sides* of *both* lines; but if the two circles lie on the same side of one of the two straight lines, and on different sides of the other straight line, then the contact of the two circles with the two straight lines is said to be unlike, or of different kinds.

Solution by J. DALE; E. FITZGERALD; H. MURPHY; *and others.*



This Question may also be enunciated as follows. If O_2, O_3 be the centres of the escribed circles, touching the sides CA, AB, respectively, of the triangle ABC; and if D_2, D_3 be the points where these circles touch BC; then the straight lines O_2D_3, O_3D_2 intersect in the middle point O of the perpendicular AP from A on BC. Let O_2D_3 meet AP in O; then $O_2D_2 : D_2D_3 = OP : PD_3$. But

$$O_2D_2 = r_2 = \frac{\Delta}{s_2},\quad D_2D_3 = s + s_1 = b + c,\quad \text{and}\quad PD_3 = s - b\cos C = \frac{(b+c)s_2}{a};$$

hence the above proportion gives $OP = \frac{\Delta}{a} = \frac{1}{2}AP$.

In like manner it may be proved that O_3D_2 also passes through O.

1698. (Proposed by MATTHEW COLLINS, B.A.)—Prove that the foot of a perpendicular on a common tangent (a) to two circles, drawn from the point of intersection of the two contrary common tangents (b and c) to those circles, lies in a straight line with the centre of one of the circles and the reflexion of the centre of the other with respect to the tangent (a).

Solution (1) *by* ABRACADABRA; (2) *by* G. O. HANLON; *and others.*

[See Figure to preceding Solution (Quest. 1697).]

1. Since $\dfrac{r_3}{O_3Z} = \dfrac{AP}{AZ} = \dfrac{r_2}{O_2Z}$, $\therefore \dfrac{r_3 - AP}{AP - r_2} = \dfrac{AO_3}{AO_2} = \dfrac{r_3}{r_2}$, whence $AP = \dfrac{2r_2r_3}{r_2+r_3}$. [This, we may observe, proves that the perpendicular from an angle on the opposite side of a triangle is an harmonic mean between the radii of the circles escribed to the other two sides, a property used in the solution of Quest. 1663.] Now $\dfrac{r_3}{r_2+r_3} = \dfrac{AO_3}{O_2O_3}$, and, if Q be the reflexion of O_2 with respect to D_2D_3, we have $2r_2 = O_2Q$; hence it follows that $AP : AO_3 = O_2Q : O_2O_3$; and therefore Q, P, O_3 are in a straight line.

2. *Otherwise:* Since $O_3A : AO_2 = O_3Z : ZO_2$, and P lies in the semicircle on AZ, therefore $O_3P : PO_2 = O_3A : AO_2$, hence $\angle O_3PA = APO_2$, and therefore $\angle O_3PD_3 = O_2PD_2 = QPD_2$; whence it follows that Q, P, O_3 are a straight line.

1756. (Proposed by W. S. BURNSIDE, B.A.)—If normals drawn to a central quadric at the points (1), (2), (3), (4), (5), (6) pass through a point, and r_{12} denote the length of the semi-diameter parallel to the chord (1, 2); prove that
$$r^2_{12} + r^2_{34} + {}^2_{56} = a^2 + b^2 + c^2,$$
where a, b, c are the semi-axes of the quadric.

Solution by the PROPOSER.

1. To construct the normals to a quadric which lie in a plane, draw a perpendicular from the pole of the plane on it, and draw also two tangent planes to the surface through this line; the points of contact of these planes are the points at which the normals should be drawn. This being so, if n_1 and n_2 be the lengths of two intersecting normals at the points (1, 2) and p_1, p_2 the central perpendiculars on the tangent planes at the same points, then $n_1p_1 + n_2p_2 = 2r^2_{12}$. For if S be the result of substituting the coordinates of the middle point of the chord in the equation of the surface; ϖ_1, ϖ_2 the perpendiculars from that point on the tangent planes; and $2C_{12}$ the length of the chord; then, plainly, $n_1\varpi_1 + n_2\varpi_2 = 2C^2_{12}$; also $C^2_{12} = r^2_{12}S$, $\varpi_1 = p_1S$, $\varpi_2 = p_2S$; hence $n_1p_1 + n_2p_2 = 2r^2_{12}$. (See Salmon's *Conics*, 4th ed., p. 227.)

2. If (x, y, z) be the coordinates of the point through which all the normals pass, we have, from the equation of the normal at (1),
$$\frac{x_1 - x}{a^{-2}x_1} = \frac{y_1 - y}{b^{-2}y_1} = \frac{z_1 - z}{c^{-2}z_1} = n_1p_1 = \lambda_1 \text{ (say)},$$
$\left(\text{as } n_1^2 = (x_1-x)^2 + (y_1-y)^2 + (z_1-z)^2, \text{ and } p_1^{-2} = a^{-4}x_1^2 + b^{-4}y_1^2 + c^{-4}z_1^2\right)$;

therefore $\quad x = x_1\left(1 - \dfrac{\lambda_1}{a^2}\right),\ y = y_1\left(1 - \dfrac{\lambda_1}{b^2}\right),\ z = z_1\left(1 - \dfrac{\lambda_1}{c^2}\right);$

but $\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} + \dfrac{z_1^2}{c^2} = 1,\ \therefore\ \dfrac{a^2x^2}{(a^2-\lambda)^2} + \dfrac{b^2y^2}{(b^2-\lambda)^2} + \dfrac{c^2z^2}{(c^2-\lambda)^2} = 1,$

or $\lambda^6 - 2\lambda^5 (a^2+b^2+c^2) + \lambda^4 L + \lambda^3 M + \lambda^2 N + \lambda R + S = 0$, where
$R = 2a^2b^2c^2 \left\{ (b^2+c^2) x^2 + (c^2+a^2) y^2 + (a^2+b) z^2 - b^2c^2 - c^2a^2 - a^2b^2 \right\}$

$S = a^4b^4c^4 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)$; &c. &c.;

whence $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 2(a^2+b^2+c^2)$.

But $\lambda_1 + \lambda_2 = 2r^2_{12} = (n_1p_1 + n_2p_2)$, by (1);

we have then finally $r^2_{12} + r^2_{34} + r^2_{56} = a^2+b^2+c^2$.

3. From the last two coefficients of the above equation of the sixth degree it is easily seen that, if (a_1, a_2, a_3) are the elliptic coordinates of the point (x, y, z) through which the normals pass, we have

$$\frac{1}{2}\sum_{r=6}^{r=} \frac{1}{n_rp_r} = \frac{1}{a^2-a_1^2} + \frac{1}{a^2-a_2^2} + \frac{1}{a^2-a_3^2}.$$

1743. (Proposed by the EDITOR.)—1. If P, Q be two points in the path of a projectile; PT, QT the tangents at P, Q; and α, β, γ the angles made with the horizon by PT, TQ, QP, respectively; prove geometrically that $\tan\alpha + \tan\beta = 2\tan\gamma$.

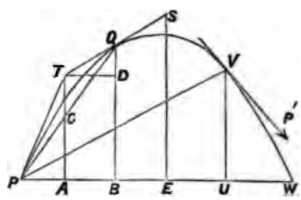
2. Hence show that if a particle be projected from one end of the horizontal base of an isosceles triangle so as exactly to shoot down the opposite side; and if α be the angle of projection, γ the angle at the base of the triangle; then $\tan\alpha = 3\tan\gamma$.

3. Prove that the velocities at P, Q are proportional to PT, QT.

Solution by J. TAYLOR; E. MCCORMICK; J. DALE; *and others.*

1. Through T, Q draw the diameters TA, QB to the horizontal line through P; also draw TD parallel to PB; and let TA meet PQ in C. Then, since the diameter through the intersection of two tangents to a parabola bisects the chord of contact, we have PC = CQ, and therefore PA = AB = TD; hence

$$\tan\alpha + \tan\beta = \frac{TA}{AP} + \frac{QD}{DT} = 2\frac{QB}{BP} = 2\tan\gamma.$$



2. Here PV, one side of the isosceles triangle PVP′, is a chord, and the other side VP′ is a tangent, to the parabolic path PVW; moreover $\tan\beta = -\tan\gamma$; therefore, by (1), we have $\tan\alpha = 3\tan\gamma$.

3. In PB produced make BE = PA = AB, and produce TQ to meet the diameter through E in S; then TQ = QS; hence, since the horizontal velocity of a projectile is uniform, the velocities at P, Q are proportional to PT, QS, that is to PT, QT.

[From (3) we obtain a simple proof of Professor SYLVESTER's theorem (Quest. 1439; *Reprint*, Vol. II., p. 55), since the velocities at P, Q are known to be as the distances of these points from the directrix, or from the focus.]

1229. (Proposed by S. WATSON.)—Show that the average area of all the triangles that can be formed by joining three points, taken *ad libitum* upon the surface of a given triangle, is one-twelfth of the triangle.

Solution by PROFESSOR SYLVESTER.

Let M be the average value of any quantity of the i^{th} order of linear magnitude depending solely on the internal constitution of a group of n elements taken arbitrarily in a triangle; let m be the average value of such quantity when three of the elements are limited respectively to the three sides, and μ_1, μ_2, μ_3, its values when one of them is limited to an angle and another to an opposite side: it may be proved that

$$M = \frac{8n(n-1)(n-2)}{(2n+i)(2n+i-1)(2n+i-2)}\, m + \Sigma \frac{4n(n-1)}{(2n+i)(2n+i-1)(2n+i-2)}\, \mu \quad \ldots\ldots (\Omega),$$

provided $2n+i-2$ is greater than zero. In the case before us, $n=3$, $i=2$, and consequently $M = \frac{1}{14}(2m+\mu_1+\mu_2+\mu_3)$. Call A the area of the triangle; then it is easily seen that $m = \frac{1}{4}A$; m, in fact, being the area of the triangle obtained by joining together the middle points of the sides of the given triangle. Again, as regards μ_1, corresponding to one point at the angle A, a second at the side BC, and a third roving anywhere over the surface of the triangle, it is easily seen that, taking any point P in BC, and putting $BC = a$, $BP = x$, we have

$$a \,.\, A \,.\, \mu_1 = \frac{A^2}{3}\int_0^a dx\, \frac{x^2+(a-x)^2}{a^2} = \frac{2aA^2}{9};$$

hence $\mu_1 = \frac{2}{9}A = \mu_2 = \mu_3$; and therefore $M = \frac{1}{14}\left(\frac{2}{4}+\frac{2}{3}\right)A = \frac{1}{12}A$.

NOTE.—This solution is offered in exemplification of the power of the general theorem marked (Ω). An analogous theorem, giving rise to a depression of *four* degrees in the integrations, applies to the tetrahedron, and in general the summations of any intrinsic affection of a group of elements, whether free or conditional, for any plane polygon may be reduced *three* and for any polyhedron *four* degrees; the condition or conditions to which the group is subject being understood to be intrinsic conditions of form, and not involving linear magnitude. Probability questions come under the head of conditional groups, and the value of i, the degree of the dimensions of the affection, is for such questions *zero*. By an *average* value of an intrinsic affection of a conditional group of elements, in the application of this method, is to be understood the aggregate of the affection divided by a suitable power of the range, or more generally by the product of the ranges of the several elements. The *mean* value in its ordinary sense will be the average value thus defined of the intrinsic affection of the conditional group *divided* by the *probability* of the fulfilment of the conditions. Any such conditions may either be algebraical; as for instance that the triangle cornered by three elements shall be obtuse, or that the ratio of no two sides shall exceed a given quantity; or it may be transcendental, as, for instance, that the quadrilateral cornered by four elements shall be convex. Provided only that each such condition be intrinsic and independent of the scale of linear magnitude, the theorems of reduction apply universally, being founded on primordial conceptions of simple and compound similitude.

1739. (Proposed by H. McColl.)—Let $f(x) = A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0$. Let $f_1(x) = A_{n-1}x^{n-1} + 2A_{n-2}x^{n-2} + \ldots + (n-1)A_1 + nA_0$. Form $f_2(x)$ from $f_1(x)$ in the same manner, and so on to $f_n(x)$. Let a and b be any two quantities *of the same sign*, of which a is greater numerically than b. Show that if the series of functions $f(x), f_1(x), f_2(x) \ldots f_n(x)$ give p changes of signs when $x = a$, and q changes when $x = b$, the number of real roots of $f(x) = 0$ between a and b cannot exceed $p-q$.

Solution by the Proposer.

Let $y = x^{-1}$, and let $\phi(y) = 0$ be the equation whose roots are the reciprocals of the roots of $f(x) = 0$. Then $f(x) = x^n \phi(y)$; and generally denoting the rth derived function of $\phi(y)$ by $\phi^r(y)$ we shall have $f_r(x) = x^{n-r}\phi^r(y)$; so that when x is positive, $f_r(x)$ and $\phi^r(y)$ have the same sign ; and when x is negative, $f_r(x)$ and $\phi^r(y)$ have the same or opposite signs according as $n-r$ is even or odd. If therefore the series of functions $f(x), f_1(x), f_2(x), \ldots f_n(x)$ give p changes of signs when $x =$ a positive quantity a, and q changes when $x =$ a positive quantity b, the series of functions $\phi(y), \phi'(y), \phi^2(y), \ldots \phi^n(y)$ will also give p changes when $y = a^{-1}$, and q changes when $y = b^{-1}$. But in this case, by Fourier's theorem, the number of real roots of $\phi(y) = 0$ between a^{-1} and b^{-1} (which is the same as the number of real roots of $f(x) = 0$ between a and b) cannot exceed $p-q$. Again, if the series of functions $f(x), f_1(x), f_2(x), \ldots f_n(x)$ give p changes of signs when $x =$ a negative quantity a, and q changes when $x =$ a negative quantity b, the series of functions $\phi(y), \phi'(y), \phi^2(y), \ldots \phi^n(y)$ will give $n-p$ changes when $y = a^{-1}$ and $n-q$ changes when $y = b^{-1}$. But in this case also, by Fourier's theorem, the number of real roots of $\phi(y) = 0$ between a^{-1} and b^{-1} (that is, the number of real roots of $f(x) = 0$ between a and b) cannot exceed $(n-q)-(n-p)$, that is, $p-q$.

N.B.—If a and b be taken sufficiently near, and the equation $f(x) = 0$ have no equal roots, this theorem will always inform us of the exact number of real roots between a and b when Fourier's theorem leaves the matter doubtful. From this, and the similarity between the two theorems, the one may be called the complement of the other.

Since $f_1(x)$ evidently $= nf(x) - xf'(x)$, whatever value we give to x, and $f'(x) = x^{-1}\left\{nf(x) - f_1(x)\right\}$, it is evident that if $f'(a)$ and $f(a)$ have opposite signs (a being positive), $f_1(a)$ and $f(a)$ will have the same sign; and, if $f_1(a)$ and $f(a)$ have opposite signs, $f'(a)$ and $f(a)$ will have the same sign. Attention to these facts will often prevent unnecessary labour in ascertaining the sign of $f'(a)$ or $f_1(a)$, when we employ this theorem and Fourier's simultaneously.

1761. (Proposed by H. McColl.)—Let $f(x) = 0$ be any equation of the nth degree. Let $f^r(x)$ be the rth derived function of $f(x)$. Let it be

known that $f^r(x) = 0$ has only one root between the two quantities a and b. Show that by giving a more general expression to the trial fraction in Horner's method of approximation, we can simultaneously approximate as nearly as we please to the root of $f^r(x) = 0$, separate any two roots of $f^{r-1}(x) = 0$ that may exist between a and b, and obtain information as to the number of roots of $f^{r-2}(x) = 0$, $f^{r-3}(x) = 0 \ldots . f'(x) = 0$, $f(x) = 0$ passed over in the operation. Show also that the same process will enable us to approximate simultaneously to any maximum or minimum value of $f(x)$, and to the value of x, which makes $f(x)$ a maximum or minimum.

Solution by the PROPOSER.

1. Let x_0 denote the root of $f^r(x) = 0$ between a and b. Let a_0 be any approximation to x_0. Assuming $x_0 = a_0 + x_1$, we get $f^r(a_0 + x_1) = f^r(a_0)$ $+ f^{r+1}(a_0)\,x_1$ + terms involving higher powers of x_1. If a_0 is a sufficiently near approximation to x_0, these terms are numerically small compared with the first two, and $x_1 = -\dfrac{f^r(a_0)}{f^{r+1}(a_0)}$ nearly.

2. Again; $f(x_0) = f(a_0 + x_1) = f_1(x_1)$ say $= A_n x_1^n + A_{n-1} x_1^{n-1} + \ldots$ $\ldots + A_1 x_1 + A_0$ say. The coefficients A_0, A_1, A_2, &c.. may be found successively by Horner's method; and we know from Taylor's theorem that $f^r(a_0) = \underline{|r}\, A_r$, and $f^{r+1}(a_0) = \underline{|r+1}\, A_{r+1}$.

But it has been shown that $x_1 = -\dfrac{f^r(a_0)}{f^{r+1}(a_0)}$ nearly;

therefore $x_1 = -\dfrac{\underline{|r}\, A_r}{\underline{|r+1}\, A_{r+1}}$ nearly; that is, $-\dfrac{A_r}{(r+1)\, A_{r+1}}$ nearly.

3. Again; assuming $x_1 = a_1 + x_2$, (a_1 being any approximation to x_1,) we get $f(x_0) = f(a_0 + a_1 + x_2) = f_1(a_1 + x_2) = f_2(x_2)$ say $= A_n x_2^n + A_{n-1} x_2^{n-2}$ $+ \ldots . + A_1 x_2 + A_0$ say. The new values of the coefficients A_0. A_1, A_2, &c., may again be found successively by Horner's method from their former values in $f_1(x_1)$, and by the same reasoning as before we have

$$x_2 = -\frac{A_r}{(r+1)\, A_{r+1}} \text{ nearly.}$$

4. If we continue this process and assume x_0 successively $= a_0 + x_1 =$ $a_0 + a_1 + x_2 = a_0 + a_1 + a_2 + x_3 =$ &c., a_m being any approximation to x_m, representing the expansion of $f_{m-1}(a_{m-1} + x_m)$ by $f_m(x_m)$, and denoting generally the coefficient of x_m^r by A_r, we shall get $-\dfrac{A_r}{(r+1)\, A_{r+1}}$ as an approximation to x_m which gets closer and closer as m increases.

5. The approximation a_m to which we are guided by $-\dfrac{A_r}{(r+1)\, A_{r+1}}$ as a trial fraction is perfectly arbitrary, and may be taken numerically

greater or less than x_m, but the latter more conveniently, as the $a_0, a_1, a_2, \ldots a_m$ will all have the same sign. The approximation a_m is numerically greater or less than x_m according as A_r changes or retains its sign in passing from its value in $f_m(x_m)$ to its value in $f_{m+1}(x_{m+1})$.

6. The limit which $f_m(x_m)$ approaches as m increases is

$$f(x_0)+f'(x_0)x_m+\frac{f''(x_0)}{\underline{|2}}x_m^2+\ldots+\frac{f^n(x_0)}{\underline{|n}}x_m^n,$$

in which $\frac{f^r(x_0)}{\underline{|r}}$, the limit to which A_r approaches, is necessarily zero, since x_0 is by hypothesis the root of $f^r(x)=0$.

7. If the equation $f^{r-1}(x)=0$ has two real and nearly equal roots between a and b, the root x_0 of $f^r(x)=0$ is situated between them: so that our approximation to x_0 is also an approximation to each of the roots of $f^{r-1}(x)=0$ till one of these roots is passed over, which will be indicated by A_{r-1} changing its sign.

8. When A_{r-1} retains its sign as it approaches and when it reaches its limit $\frac{f^{r-1}(x_0)}{\underline{|r-1}}$, the equation $f^{r-1}(x)=0$ cannot have two real roots between a and b; and we may be sure that this is the case as soon as we find that $rA_rb_m+A_{r-1}$ has the same sign as A_{r-1}. b_m being a numerically superior approximation to x_m obtained by the trial fraction.

9. When $f^{r-1}(x)=0$ has two equal roots between a and b, the limit to which A_{r-1} approaches, as well as the limit to which A_r approaches, is zero.

10. And generally if $f^{r-s}(x)=0$ has $s+1$ real and nearly equal roots between a and b, the root x_0 of $f^r(x)=0$ is situated between the greatest and least; so that our approximation to x_0 is also an approximation to each of the $s+1$ roots of $f^{r-s}(x)=0$ till p roots are passed over, which will be indicated by a decrease or increase (according as a_m was positive or negative) of p signs in those given by the series of coefficients $A_{r+1}, A_r, A_{r-1}, \ldots A_{r-s}$.

11. If these $s+1$ roots are equal, all the coefficients $A_r, A_{r-1}, A_{r-2}, \ldots A_{-s}$ approach the limit zero.

12. If when we know that no more than $s+1$ real roots of $f^{r-s}(x)=0$ exist between a and b, the coefficients $A_r, A_{r-1}, A_{r-2}, \ldots A_{r-(s-1)}$ approach the limit zero, while A_{r-s} without changing its sign approaches a limit which is *not* zero; then $f^{r-s}(x)=0$ cannot have two real roots between a and b; and if we find that $f^{r-s}(x)=0$ has no real root between a and b, then $f^{r-s-1}(x)=0$ cannot have two real roots in the same interval.

13. When from the rapid numerical decrease of $A_r, A_{r-1}, A_{r-2}, \ldots A_1, A_0$ we suspect that $f(x)=0$ has $r+1$ equal roots each $=x_0$, the follow-

ing tests will immediately remove all uncertainty on the point, provided that A_n, A_{n-1}, ... A_1, A_0, the coefficients in $f(x)$, are integers.

First, when there is only one doubtful interval. Let c be any root of $f(x)=0$ repeated q times. Then $A_n c$ is a whole number, and $(A_n x - A_n c)\,M^{-1}$, in which M is the greatest common measure of A_n and $A_n c$, will divide $f(x)$ q times.

Next, when there are two doubtful intervals. If c and c_1 be two roots of $f(x)=0$, each of which is repeated q times; then $A_n\,(x-c)\,(x-c_1)\,M^{-1}$, that is $\left\{A_n x^2 - A_n(c+c_1)\,x + A_n cc_1\right\} M^{-1}$, in which M is the greatest common measure of the *integral* coefficients A_n, $A_n(c+c_1)$, $A_n cc_1$, will divide $f(x)$ q times.

The same principle may evidently be extended to any number of doubtful intervals.

14. When $f(x)$ has a maximum or minimum value; since x_0 is a real root of $f^r(x)=0$, and the limit to which A_0 approaches as m increases is $f(x_0)$, by Art. 6, it is evident that $f(x_0)$ will, if we give the proper value to r, be a maximum or minimum value of $f(x)$.

On the Numerical Solution of Equations.

(Continued from the Solutions of Questions 1739, 1761.)

Form Fourier's functions and the functions given in Quest. 1739. and apply both theorems to the equation simultaneously. If both leave us in doubt as to the exact number of real roots between $a\,.\,10^n$ and $(a+1)\,10^n$, in which a is zero or some integer not greater numerically than 9, and n is any integer positive, negative, or zero, apply the process given in Question 1761. Take for example the equation $2x^4-9x^3+19x^2-21x+8=0$.

Fourier's Functions.	*Complementary Functions.*
$2x^4-9x^3+19x^2-21x+8$	$2x^4-9x^3+19x^2-21x+8$
$8x^3-27x^2+38x-21$	$-9x^3+38x^2-63x+32$
$24x^2-54x+38$	$38x^2-126x+96$
$48x-54$	$-126x+192$
$+$	$+$

Fourier's functions give 4 changes when $x=-\infty$, 4 changes when $x=0$, 3 changes when $x=1$, 0 change when $x=2$, and none when $x=\infty$. The complementary functions give 0 change when $x=-\infty$, 0 change when $x=0$, 1 change when $x=1$, 2 changes when $x=2$, and 4 changes when $x=\infty$. Both theorems, therefore, inform us that no real root exists between $-\infty$ and 0, and only one real root between 0 and 1. The complementary theorem informs us that only one root exists between 1 and 2; and Fourier's theorem informs us that no real root exists greater than 2. The equation, therefore, has only two real roots, one between 0 and 1, and the other between 1 and 2; and each may be approximated to in the usual way by Horner's method.

Next, take the equation $x^4-12x^3+13x^2+24x-30=0$. Either of the theorems will inform us that only one real root exists between $-\infty$ and 0, none between 0 and 1, and only one between 2 and ∞; but both leave us in doubt as to whether two real roots or none exist between 1 and 2. We

must, therefore, apply the process of Quest. 1761 to separate the two roots if they are real.

We see by Fourier's theorem that $f'(x) = 0$ has one real root between 1 and 2. Denoting this root by x_0 we assume $x_0 = 1 + x_1$ and expand $f(1+x_1)$ by Horner's method, the result (writing coefficients only) being

$$1-8-17+18-4 = f_1(x_1).$$

Our trial fraction for x_1 (since $r = 1$) is $-\frac{A_1}{2A_2}$ that is $\frac{18}{2 \times 17}$, which is between $\cdot 5$ and $\cdot 6$. But if we assume $x_1 = \cdot 5 + x_2$ and expand $f_1(\cdot 5 + x_2)$ we shall find that A_1 as the coefficient of x_2 is negative, its value 18 as the coefficient of x_1 having been positive. This shows that $\cdot 5$ is greater than x_1; so to avoid a negative value of x_2 we assume $x_1 = \cdot 4 + x_2$ and expand $f_1(\cdot 4 + x_2)$, the result being

$$1-6\cdot 4-25\cdot 64+\cdot 816-\cdot 0064 = f_2(x_2).$$

Our trial fraction for x_2 is $\frac{\cdot 816}{2 \times 25\cdot 64}$ which is between $\cdot 01$ and $\cdot 02$. We therefore assume $x_2 = \cdot 01 + x_3$ and expand $f_2(\cdot 01 + x_3)$, the result being

$$1-6\cdot 36-25\cdot 8314+\cdot 301284-\cdot 00081039 = f_3(x_3).$$

The trial fraction for x_3 is $\frac{\cdot 301284}{2 \times 25\cdot 8314}$ which is between $\cdot 005$ and $\cdot 006$. We therefore assume $x_3 = \cdot 005 + x_4$ and expand $f_3(\cdot 005 + x_4)$. Before completing the expansion, however, we find at the end of the first line of Horner's process, that the absolute term A_0, or the coefficient of x_4^0, is positive, while its value as the absolute term or coefficient of x_3^0, in the former expansion was negative. This shows that a root of $f(x) = 0$ has been passed over, and that, consequently, $f(x) = 0$ has one root between $1\cdot 41$ and $1\cdot 415$, and another between $1\cdot 415$ and $1\cdot 42$.

If now we remove the restrictions placed upon x_1, x_2, x_3, and make them variables, the equation $f_3(x_3) = 0$ is the equation whose roots are less by $1\cdot 41$ than those of the original equation $f(x) = 0$; so that if we wish to approximate more closely to either of the two roots which we have just separated, we have only to continue the operation—merely changing the trial fraction from $-\frac{A_1}{2A_2}$ to $-\frac{A_0}{A_1}$. A brief examination of the coefficients A_2, A_1, A_0 suffices to show that the smallest positive root of $f_3(x_3) = 0$ is between $\cdot 004$ and $\cdot 005$, and the next between $\cdot 007$ and $\cdot 008$; so that the first four figures of the two roots of $f(x) = 0$ are respectively $1\cdot 414$ and $1\cdot 417$.

Next, take the equation $x^4 - 12x^3 + 13x^2 + 26x - 33 = 0$. As in the last example, the interval between 1 and 2 is doubtful; so we approximate to $1\cdot 45$ &c., the root of $f'(x) = 0$, the second expansion $f_2(x_2)$ being $x_2^4 - 6\cdot 4x_2^3 - 25\cdot 64x_2^2 + 2\cdot 816x_2 - \cdot 2064$. The trial fraction $\frac{2\cdot 816}{2 \times 25\cdot 64}$ shows clearly that x_2 is situated between $\cdot 05$ and $\cdot 06$; and since $\cdot 06 \times 2\cdot 816 - \cdot 2064$ has the same sign as $-\cdot 2064$, we are sure, without carrying the process farther, that (Quest. 1761, Art. 8) A_0 will not change its sign in approaching its limit, and that, consequently, the roots between 1 and 2 are imaginary.

Lastly, take the equation $49x^4 - 357x^3 + 1002x^2 - 1305x + 675 = 0$. A doubtful interval exists between 2 and 3, and approximating by the process of Quest. 1761 to $2\cdot 14285$ &c., the root of $f'(x) = 0$, we find that not only A_1 but apparently also the absolute term A_0 approaches the limit zero. This leads us to suspect that $2\cdot 1428$ &c. is not only a root of $f'(x) = 0$, but also

a repeated root of $f(x) = 0$. Applying, therefore, the tests given in Quest 1761, Art. 13, we find that $49(x-2{\cdot}1428) = 49x-105$ nearly, which, striking out the factor 7, becomes $7x-15$, a factor which will twice divide $f(x)$. Therefore, $2{\cdot}1428\ldots$, or $2\frac{1}{7}$, is a repeated root of $f(x) = 0$.

1812. (Proposed by Professor CAYLEY.)—Find the envelope of a series of circles which touch a given straight line and have their centres in the circumference of a given circle. [See Quest. 1771.]

Solution by the REV. J. L. KITCHIN, M.A.

Let $x^2+y^2 = r^2 \ldots (1)$ be the equation of the given circle, and $x = c$ that of the given line; then the equation of any circle of the system is

$$(X-x)^2+(Y-y)^2 = (c-x)^2, \text{ or } X^2-2xX+(Y-y)^2 = c^2-2cx \ldots (2);$$

where x, y are the parameters subject to the condition (1).

Differentiating (1) and (2), and equating the two values of $\dfrac{dy}{dx}$, we get

$$\frac{X-c}{Y-y} = \frac{x}{y} \ldots (3); \quad \therefore \frac{(X-c)^2+(Y-y)^2}{(X-c)^2} = \frac{r^2}{x^2} = \frac{2(x-c)}{X-c}, \text{ by } (2);$$

$$\therefore r^2(X-c) = 2x^3-2cx^2; \text{ whence, putting } x = r\cos\theta, y = r\sin\theta,$$

$$X = c+2r\cos^3\theta-2c\cos^2\theta = \tfrac{3}{2}r\cos\theta-c\cos 2\theta+\tfrac{1}{2}r\cos 3\theta \ldots (4).$$

Again, from (3) with use of (4), we get

$$Y\cos\theta = r\sin\theta\cos\theta-c\sin\theta+\tfrac{3}{2}r\sin\theta\cos\theta-c\cos 2\theta\sin\theta+\tfrac{1}{2}r\cos 3\theta\sin\theta,$$

therefore $\quad Y = \tfrac{3}{2}r\sin\theta-c\sin 2\theta+\tfrac{1}{2}r\sin 3\theta \quad \ldots\ldots\ldots\ldots (5).$

From (4) and (5), $2X\cos\theta+2Y\sin\theta = 3r+r\cos 2\theta-2c\cos\theta \ldots\ldots (6)$,

also $\quad (2X-3r\cos\theta)^2+(2Y-3r\sin\theta)^2 = r^2+4c^2-4cr\cos\theta \quad \ldots\ldots (7).$

From (6) and (7), $\quad (r\cos\theta)^2-\tfrac{4}{3}c(r\cos\theta) = \tfrac{1}{3}(X^2+Y^2-r^2-c^2)$,

therefore $r\cos\theta = x = \tfrac{1}{3}\left\{2c+\sqrt{3(X^2+Y^2-r^2)+c^2}\right\} = \tfrac{1}{3}(2c+R)$, say;

whence $\quad 27r^2(X-c) = 2(2c+R)^3-6c(2c+R)^2 = 2R^3+6cR^2-8c^3$,

therefore $\quad 27r^2(X-c)+2c^3-18c(X^2+Y^2-r^2) = 2R^3$,

or $\quad 4\left\{3(X^2+Y^2-r^2)+c^2\right\}^3 = \left\{27r^2(X-c)+2c^3-18c(X^2+Y^2-r^2)\right\}^2$,

the equation to the envelope required, and which is of the sixth degree.

If the whole be orthogonally projected on a plane passing through the axis of x, the given and the variable circles become similar ellipses; hence if a, b be the semi-axes of the given circle so projected into an ellipse, by writing in the last equation x for X and $ab^{-1}y$ for Y, we shall obtain the envelope of a series of ellipses which touch a given straight line at the extremities of their major axes, and have their centres on another ellipse, to which the variable ellipse is similar and similarly placed.

1814. (Proposed by A. F. TORRY, M.A.)—A triangle is inscribed in an ellipse and envelopes a confocal ellipse: prove that the points of contact lie on the escribed circles.

Solution by E. FITZGERALD; J. DALE; *and others.*

If ABC be the triangle, (the figure is easily imagined,) the tangents B'C', C'A', A'B' to the outer ellipse at A, B, C are the external bisectors of the angles CAB, ABC, BCA; moreover, if A'D, B'E, C'F be drawn perpendicular to BC, CA, AB respectively, D, E, F will be the points of contact of the inscribed confocal ellipse; and D, E, F are obviously the points of contact of the escribed circles with the sides BC, CA, AB.
[See Salmon's *Conics*, 4th ed., Art. 189, and Art. 2:6, Ex. 3]

1816. (Proposed by R. BALL, M.A.)—Express the roots of the equation

$$(ae-4bd+3c^2)(ax^4+4bx^3+6cx^2+4dx+e)^2-3\left\{(ac-b^2)x^4+2(ad-bc)x^3\right.$$
$$\left.+(ae+2bd-3c^2)x^2+2(be-cd)x+(ce-d^2)\right\}^2=0,$$

in terms of the roots $\alpha, \beta, \gamma, \delta$ of $x^4+4bx^3+6cx^2+4dx+e=0$.

Solution by PROFESSOR CAYLEY.

Writing

$$U=(a,b,c,d,e)(x,y)^4=a(x-\alpha y)(x-\beta y)(x-\gamma y)(x-\delta y)$$
$$H=\left(ac-b^2,\ \tfrac{1}{2}(ad-bc),\ \tfrac{1}{6}(ae+2bd-3c^2),\ \tfrac{1}{2}(be-cd),\ ce-d^2\right)(x,y)^4$$
$$I=ae-4bd+3c^2;$$

then, considering $x:y$ as the unknown quantity, it is required to find the roots of the equation $IU^2-3H^2=0$ in terms of the roots $(\alpha, \beta, \gamma, \delta)$ of the equation $U=0$; or, what is the same thing, it is required to find the linear factors of the function IU^2-3H^2. The function in question is the product of four quadratic factors, rational functions of $(\alpha, \beta, \gamma, \delta)$; and these being known, the four pairs of linear factors can be determined each of them by the solution of a quadratic equation. In fact, writing

$$\Theta_\alpha=a\left\{(\beta-\alpha)(x-\gamma y)(x-\delta y)+(\gamma-\alpha)(x-\delta y)(x-\beta y)\right.$$
$$\left.+(\delta-\alpha)(x-\beta y)(x-\gamma y)\right\},$$

$$\Theta_\beta=a\left\{(\gamma-\beta)(x-\delta y)(x-\alpha y)+(\delta-\beta)(x-\alpha y)(x-\gamma y)\right.$$
$$\left.+(\alpha-\beta)(x-\gamma y)(x-\delta y)\right\},$$

$$\Theta_\gamma=a\left\{(\delta-\gamma)(x-\alpha y)(x-\beta y)+(\alpha-\gamma)(x-\beta y)(x-\delta y)\right.$$
$$\left.+(\beta-\gamma)(x-\delta y)(x-\alpha y)\right\},$$

$$\Theta\ =a\left\{(\alpha-\delta)(x-\beta y)(x-\gamma y)+(\beta-\delta)(x-\gamma y)(x-\alpha y)\right.$$
$$\left.+(\gamma-\delta)(x-\alpha y)(x-\beta y)\right\},$$

we have identically $256\,(IU^2-3H^2) = \Theta_\alpha\Theta_\beta\Theta_\gamma\Theta_\delta$; so that the quadratic factors of IU^2-3H^2 are Θ_α, Θ_β, Θ_γ, Θ_δ. To show that this is so, it is to be remarked that the product $\Theta_\alpha\Theta_\beta\Theta_\gamma\Theta_\delta$ is a symmetrical function of the roots $\alpha, \beta, \gamma, \delta$, and consequently a rational and integral function of the coefficients (a, b, c, d, e) of U; moreover Θ_α, Θ_β, Θ_γ, Θ_δ being each of them a covariant (an irrational one) of U, the product in question must be a covariant. But a covariant is completely determined when the leading coefficient is given; hence it will be sufficient to show that the leading coefficients, or coefficients of x^8, in the functions $\Theta_\alpha\Theta_\beta\Theta_\gamma\Theta_\delta$ and $256(IU^2-3H^2)$ are equal to each other. Writing for a moment $\Sigma\alpha = p$, $\Sigma\alpha\beta = q$, $\Sigma\alpha\beta\gamma = r$, $\alpha\beta\gamma\delta = s$, the coefficient of x^2 in $a^{-1}\Theta_\alpha$ is $\beta+\gamma+\delta-3\alpha$, which $= p-4\alpha$; we have thence the product $(p-4\alpha)\,(p-4\beta)\,(p-4\gamma)\,(p-4\delta)$, which is $=p^4-4p^3.p+16p^2.q-64p.r+256s$, $=256s-64pr+16p^2q-3p^4$.

Hence, restoring the omitted factor a^4, and observing that we have $p=-\dfrac{4b}{a}$, $q=\dfrac{6c}{a}$, $r=-\dfrac{4d}{a}$, $s=\dfrac{e}{a}$, the coefficent of x^8 in $\Theta_\alpha\Theta_\beta\Theta_\gamma\Theta_\delta$ is

$$256\,(a^3e-4a^2bd+6ab^2c-3b^4),\text{ or } 256\left\{(ae-4bd+3c^2)\,a^2-3\,(ac-b^2)^2\right\},$$

and is consequently equal to the coefficient of x^8 in $256\,(IU^2-3H^2)$; which proves the theorem.

It may be remarked that the leading coefficient of IU^2-3H^2 is $=a^{-1}\,(a, b, c, d, e)\,(b, -a)^4$; and that for a quantic $U=(a, b, \ldots)\,(x, y)^n$ of the order n we have a corresponding covariant of the order $n(n-2)$, the leading coefficient of which is $=a^{-1}\,(a, b, \ldots\ldots)\,(b, -a)^n$. For $n=2$, this is the invariant (discriminant) $ac-b^2$; for $n=3$ it is the cubicovariant $(a^2d-3abc+2b^3, \ldots\ldots\ldots)\,(x, y)^3$; for $n=4$ it is, as we have seen, the covariant IU^2-3H^2. For $n=5$, the leading coefficient $a^4f-5a^3be+10a^2b^2d-10ab^3c+4b^5$ is $=a^2\,(a^2f-5abe+2acd+8b^2d-6bc^2)-2\,(ac-b^2)\,(a^2d-3abc+2b^3)$, which shows that the covariant in question (of the order 15) is $=U^2$ (No. 17)-2 (No. 15) (No. 18), where the Nos. refer to the Tables of my Second Memoir on Quantics, *Phil. Trans.*, vol. 146 (1856), pp. 101—126.

[The roots of $\Theta_\alpha = 0$ are readily found to be

$$\frac{\alpha\,(\beta+\gamma+\delta)-(\gamma\delta+\delta\beta+\beta\gamma)\pm\left\{\tfrac{1}{3}\left[(\alpha-\beta)^2\,(\gamma-\delta)^2+(\alpha-\gamma)^2\,(\delta-\beta)^2+(\alpha-\delta)^2\,(\beta-\gamma)^2\right]\right\}^{\frac{1}{2}}}{3\alpha-(\beta+\gamma+\delta)}$$

these then, with three similar pairs, express the eight roots as required.]

1823. (Proposed by W. K. CLIFFORD.)—The conicoids which pass through six fixed points in space, intersect any plane in a series of conics having a common self-conjugate quadrilateral. *Any four* conics have a common self-conjugate quadrilateral. (DEF.—A quadrilateral is *self-conjugate* in respect of a conic which divides its diagonals harmonically.)

Solution by PROFESSOR CREMONA.

On donne, dans un plan, quatre coniques (1), (2), (3), (4). Les trois premières déterminent un réseau, dont la courbe Hessienne (du 3e ordre) est le lieu des couples de points conjugués par rapport à ces coniques (1), (2), (3). De même, les coniques (1), (2), (4) donnent une autre cubique. Ces deux cubiques passent évidemment par les sommets du triangle conjugué à (1) et (2); donc elles se couperont en six autres points. Si o est l'un de ces points, les droites polaires de o par rapport aux quatre coniques se rencontreront en un même point o' qui appartiendra par conséquent aux deux cubiques. Donc, les six points communs aux deux cubiques sont conjugués, deux à deux, par rapport aux quatre coniques; et, par suite d'un théorème connu (dû à M. Hesse), ils sont les sommets d'un quadrilatère complet. Ainsi, il y a un quadrilatère dont les diagonales sont divisées harmoniquement par les quatre coniques. De plus, cette propriété se vérifie évidemment pour toutes les coniques liées linéairement aux coniques données.

On peut obtenir un tel système linéaire des coniques

$$(1) + \lambda\,(2) + \mu\,(3) + \nu\,(4)$$

en coupant par un plan un système analogue de surfaces du second ordre, *par ex.*, le système des conicoïdes qui passent par six points donnés, ou bien le système des surfaces polaires des points de l'espace par rapport à une surface cubique, &c.

1828. (Proposed by R. WARREN, B.A.)—Exhibit the expression

$$V \equiv (\beta-\gamma)^2\,(\beta+\gamma-2\alpha) + (\gamma-\alpha)^2\,(\gamma+\alpha-2\beta) + (\alpha-\beta)^2\,(\alpha+\beta-2\gamma)$$

(1) as the product of three factors, (2) as the sum of two cubes.

Solution by W. SPOTTISWOODE, F.R.S.; Rev. J. L. KITCHIN, M.A.; W. H. LAVERTY; H. M. TAYLOR, B.A.; *and the* PROPOSER.

Let $\beta+\gamma-2\alpha = p,\ \ \gamma+\alpha-2\beta = q,\ \ \alpha+\beta-2\gamma = r,$ then

$$3\,(\beta-\gamma) = r-q,\ \ 3\,(\gamma-\alpha) = p-r,\ \ 3\,(\alpha-\beta) = q-p,\ \ p+q+r = 0;$$

therefore $V = \frac{1}{9}\left\{p\,(q-r)^2 + q\,(r-p)^2 + r\,(p-q)^2\right\}$

$$= \tfrac{1}{9}\left\{qr\,(q+r) + rp\,(r+p) + pq\,(p+q) - 6pqr\right\} = -pqr.$$

Moreover the developed form of the expression, viz.,

$$V = 2\,(\alpha^3+\beta^3+\gamma^3) - 3\,(\beta^2\gamma+\beta\gamma^2+\gamma^2\alpha+\gamma\alpha^2+\alpha^2\beta+\alpha\beta^2) - 12\alpha\beta\gamma,$$

shows almost by inspection that the form of the two cubes must be

$$(\alpha+\omega\beta+\omega^2\gamma)^3 + (\alpha+\omega^2\beta+\omega\gamma)^3,$$

where ω is an imaginary cube root of unity.

[Since $p^3+q^3+r^3-3pqr \equiv \Sigma p\,(\Sigma p^2-\Sigma qr) = 0,\ \because \Sigma p = 0,$ we have $V = -pqr = -\frac{1}{3}(p^3+q^3+r^3)$, which exhibits V as the sum of *three* cubes.]

1847. (Proposed by W. S. BURNSIDE, B.A.)—Prove that the method given by Tschirnhausen for removing the second and fourth terms from the binary quartic $(a, b, c, d, e)(x, y)^4 = 0$ depends ultimately on the solution of the equation $\lambda^3 - I\lambda + 2J = 0$,
where $I = ae - 4bd + 3c^2$, and $J = ace + 2bcd - ad^2 - eb^2 - c^3$.

Solution by ROBERT BALL, M.A.

Removing the second term from $(a, b, c, d, e)(x, 1)^4 = 0$, it becomes (putting $a=1$), $x^4 - 6Hx^2 + 4Gx + I - 3H^2 = 0$,
where $H = b^2 - ac$, $G = 2b^3 - 3abc + a^2d$, and $G^2 = 4H^3 - IH - J$.

Now the principle of Tschirnhausen's method assumes $y = x^2 + Bx + C$, and x is then eliminated between the equations

$$x^4 - 6Hx^2 + 4Gx + I - 3H^2 = 0,\ x^2 + Bx + C - y = 0;\ \text{the result being}$$

$$(C-y)^4 + 12H(C-y)^3 + (2I + 30H^2 + 12BG - 6HB^2)(C-y)^2$$
$$+\left\{28H^3 - 4IH - 16J + 24HGB + (12H^2 - 4I)B^2 - 4GB^3\right\}(C-y)$$
$$+ I^2 - 6IH^2 + 9H^4 + (12GH^2 - 4GI)B + (18H^3 - 6IH)B^2 + (I - 3H^2)B^4 = 0.$$

We must express that the coefficients of y^3 and y vanish; hence

$$-4C - 12H = 0, \text{ and } -4C^3 - 36HC^2 - 2C(2I + 30H^2 + 12BG - 6HB^2)$$
$$-28H^3 + 4IH + 16J - 24HGB + (4I - 12H^2)B^2 + 4GB^3 = 0.$$

From the first we obtain $C = -3H$, and this substituted in the second gives $GB^3 + B^2(I - 12H^2) + 12HGB - 4G^2 = 0$; and if in this last equation we change B into $\dfrac{2G}{H+\lambda}$, it becomes $\lambda^3 - I\lambda + 2J = 0$.

NOTE.—The more general investigation for the case when the elimination is performed between a cubic and a biquadratic will shortly appear in the *Quarterly Journal of Mathematics*, and various reducing cubics resulting from the process will be reduced to the form $x^3 - Ix + 2J = 0$. The present Question will be also presented in a different form.

1811. (Proposed by Professor SYLVESTER.)—(α). Prove that the chance of three points taken at random within a circle or sphere forming the vertices of a triangle of any prescribed form is the same as if two of the points are restrained to move on the circumference or surface, with a likelihood varying as the square of the distance between them.

(β). By the aid of this principle show that the chance of a triangle whose angular points are taken at random within a circle being acute is

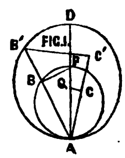
$$\int_0^{\frac{1}{2}\pi}(3\sin^3\theta\cos\theta + \theta\sin^2\theta - \tfrac{1}{2}\pi\sin^4\theta)\,d\theta \div \int_0^{\frac{1}{2}\pi}\pi\sin^2\theta\,d\theta, \text{ or } \frac{4}{\pi^2} - \frac{1}{8};$$

and that the chance of a triangle whose angular points are taken at random within a sphere being obtuse is

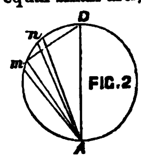
$$\int_0^{\frac{1}{2}\pi} \sin^3\theta \cos\theta \,(6-6\sin\theta-3\cos\theta+4\sin^3\theta+\cos^3\theta)\,d\theta, \quad \text{or } \frac{37}{70}.$$

(γ). Find also the chance that no angle of the triangle will exceed any magnitude α between $\frac{1}{3}\pi$ and π, distinguishing between the case where α is greater and that where it is less than $\frac{1}{2}\pi$.

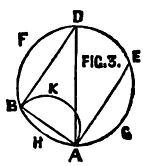
Solution by J. D. EVERETT, D.C.L.



1. Of the three points A, B, C, let A be that which is farthest from the centre of the circle. Then A may (as shown in the Solution of Quest. 1333), be made a fixed point on the circumference. To AB, AC (Fig. 1) let perpendiculars BP, CQ be drawn, meeting the diameter AD in P, Q; and let AP be greater than AQ. Then if a circle be drawn on AP as diameter, B will be on its circumference, and C within it. Hence there is no loss of generality in restricting C to lie within the variable circle which passes through A, B, and has its centre in AD. If AB be produced to meet the circumference of the given circle in B′, and AC be produced, in the same ratio, to C′, the triangle AB′C′ will be similar to ABC. Hence B may be restricted to lie on the circumference provided allowance be made for the relative probability of different directions of AB. Now if m and n (Fig. 2) be two equal small arcs, and if sectors Am, An be formed by joining A to their extremities, the relative probabilities of B being within either of these sectors are as the areas of the sectors, that is ultimately as the squares of the distances Am, An. Hence, in restricting B to lie on the circumference, we must make the likelihood vary as the square of the distance from A. This proves the theorem (α).



2. To find the probability of the triangle ABC being acute for any given position of B on the circumference, draw (Fig. 3) chords AE, BD perpendicular to AB; and on AB describe the semicircle AKB on the side next the centre. The triangle will be acute if C lies within the portion AKBDE of the circle, and obtuse if it lies in any other position. Hence the probability of an acute triangle is (AKBDE $\div$ circle). Let θ denote the angle BDA; then, taking the radius of the circle for unit of length, the area of AKBDE is $3\sin\theta\cos\theta+\theta-\frac{1}{2}\pi\sin^2\theta$, and the area of the circle is π; hence, for a given position of B, the probability of an acute triangle $(3\sin\theta\cos\theta+\theta-\frac{1}{2}\pi\sin^2\theta)\div\pi$. But the likelihood of any value of θ is as the square of AB, that is, as $\sin^2\theta$; hence the chance of the triangle being acute is



$$\int_0^{\frac{1}{2}\pi}(3\sin^3\theta\cos\theta+\theta\sin^2\theta-\tfrac{1}{2}\pi\sin^4\theta)\,d\theta \div \int_0^{\frac{1}{2}\pi}\pi\sin^2\theta\,d\theta = \frac{4}{\pi^2}-\frac{1}{8}.$$

The integrals are easily evaluated, and need no remark.

3. In adapting the foregoing reasoning to a sphere, BP and CQ (Fig. 1) will represent lines perpendicular to AB, AC, meeting the diameter AD of the sphere in P, Q; then the sphere whose diameter is AP will have B on its surface and C within it, and it may consequently be shown as before that B may be restricted to lie on the surface according to a certain law of likelihood. To find this law, let m (Fig. 2) denote the area of a small portion of the surface, then the volume of the cone whose base is m and vertex A is $\frac{1}{3}mh \sin\theta$, h denoting the distance Am and θ the angle ADm which is equal to the obliquity of the cone or the inclination of Am to a tangent plane at m. But $\sin\theta$ evidently varies as h; hence if equal small portions m be taken on different parts of the surface, the likelihood of B being in any one of them must vary as $\sin^2\theta$ or as h^2, that is, as the square of the distance from A. To find the probability of an acute triangle for any given position of B, draw (Fig. 3) through A, B planes AE, BD perpendicular to AB, and through AB draw a plane perpendicular to the plane ABDE; also on the section of the sphere made by the plane last drawn as base, describe the hemisphere AKB. The triangle will be acute if C be within the portion AKBDE of the sphere contained between the hemisphere AKB and the two planes AE, BD; and it will be obtuse if C lies in one of the other portions. Hence the chance of an obtuse triangle is the sum of these other portions divided by the volume of the sphere. Let θ denote the angle ADB; then (the radius of the sphere being unity) the volumes of the segment AHB; of the two equal segments BFD, AGE; of the hemisphere AKB; and of the whole sphere; are, respectively,

$$\tfrac{1}{3}\pi(2-3\cos\theta+\cos^3\theta),\ \tfrac{2}{3}\pi(2-3\sin\theta+\sin^3\theta),\ \tfrac{2}{3}\pi\sin^3\theta,\ \tfrac{4}{3}\pi;$$

hence, for a given position of B, the chance of an obtuse triangle is

$$\tfrac{1}{3}\pi\left\{(4-6\sin\theta+2\sin^3\theta)+(2-3\cos\theta+\cos^3\theta)+2\sin^3\theta\right\} \div \tfrac{4}{3}\pi.$$

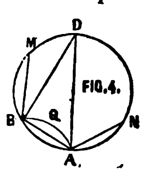
In supposing B to move over the surface, imagine the surface to be divided into narrow zones of equal width, having A as their common pole. Then the area of any zone will be proportional to its radius which is $2\sin\theta\cos\theta$; hence if all situations on the sphere were equally probable, the likelihood of any value of θ would vary as $\sin\theta\cos\theta$; and since the likelihood of different situations varies as $\sin^2\theta$, it follows by compounding ratios that the likelihood of any value of θ is as $\sin^3\theta\cos\theta$; hence, in this case, the chance of an obtuse triangle is

$$\frac{1}{4}\int_0^{\frac{1}{2}\pi} \sin^3\theta\cos\theta\,(6-6\sin\theta-3\cos\theta+4\sin^3\theta+\cos^3\theta)\,d\theta \div \int_0^{\frac{1}{2}\pi} \sin^3\theta\cos\theta\,d\theta,$$

and since the divisor here is equal to $\frac{1}{4}$, the required chance becomes

$$\int_0^{\frac{1}{2}\pi} \sin^3\theta\cos\theta\,(6-6\sin\theta-3\cos\theta+4\sin^3\theta+\cos^3\theta)\,d\theta = \frac{37}{70}.$$

[4. To solve the problem (γ) for the *circle* (and the solution for the *sphere* may be obtained in the same way) we may proceed as follows. Construct Figs. 4, 5, 6, so that $\alpha = \angle$ ABM = BAN = $\angle$ in seg. AQB. Then if $\alpha > \frac{1}{2}\pi$, Fig. 4 applies; and no angle of the triangle will exceed α when C falls within the area ANMBQA ($=\Delta_1$ suppose); moreover, when $\theta > \pi - \alpha$ the lines AN, BM lie outside the segment ADB, and then the like *area locus* of C is $\pi - 2$ seg. AQB ($=\Delta_2$ suppose). Now putting, for shortness' sake F (ϕ) to denote the area of a circular segment,



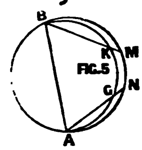
to radius unity, which contains an angle ϕ, so that $F(\phi) \equiv \pi - \phi + \sin\phi \cos\phi$, we readily find radius of arc AQB $= \sin\theta \operatorname{cosec} a$ ($= r$ suppose),
also $\Delta_1 = F(\theta) - 2F(a+\theta) - r^2 F(a)$, and $\Delta_2 = \pi - 2r^2 F(a)$.

Furthermore, for an assigned direction of AB (that is to say, for each value of θ) the measure of the number of points B projected to the circumference in Fig. 1 is $\sin^2\theta$. Hence the whole probability $p(a)$ that no angle of the triangle exceeds a magnitude a greater than $\frac{1}{2}\pi$, is

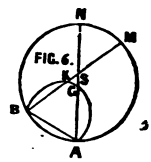
$$p(a) = \left\{\int_0^{\pi-a} \Delta_1 \sin^2\theta\, d\theta + \int_{\pi-a}^{\pi} \Delta_2 \sin^2\theta\, d\theta\right\} \div \int_0^{\frac{1}{2}\pi} \pi \sin^2\theta\, d\theta$$

$$= \frac{1}{\pi^2}\left\{\pi^2 - 3(\pi-a)^2 + \sin^2 a(\sin^2 a + 3) - \right.$$

$$\left. \tfrac{3}{2}\operatorname{cosec}^2 a\, F(a)\, F(\pi - a) + \sin a \cos a\, F(a)\right\}.$$



5. When $a < \frac{1}{2}\pi$, no angle of the triangle will exceed a if C fall within the area GKS ($= \Delta_3$) in Fig. 6, or GKMN ($= \Delta_4$) in Fig. 5; moreover Fig. 6 applies till S falls on the given circle, that is from $\theta = 0$ to $\theta = \pi - 2a$; also Fig. 5 applies from $\theta = \pi - 2a$ to $\theta = a$, when the arc AGKB coincides with the given circle; and from $\theta = a$ to $\theta = \frac{1}{2}\pi$ no such triangle can be formed.



Also $\Delta_3 = \tan a \sin^2\theta - r^2\left\{F(a) - 2F(2a)\right\}$,

and $\Delta_4 = \left\{F(\theta) - 2F(a+\theta)\right\} - r^2\left\{F(a) - 2F(2a)\right\}$.

Hence the probability that no angle of the triangle exceeds a magnitude a less than $\frac{1}{2}\pi$, is

$$p(a) = \left\{\int_0^{\pi-2a} \Delta_3 \sin^2\theta\, d\theta + \int_{\pi-2a}^{a} \Delta_4 \sin^2\theta\, d\theta\right\} \div \int_0^{\frac{1}{2}\pi} \pi \sin^2\theta\, d\theta$$

$$= \frac{1}{\pi^2}\left\{(3a-\pi)^2 + \sin^2 a(\sin^2 a + 3) - \sin^2 2a(\sin^2 2a + 3)\right.$$

$$+ \left(\tfrac{3}{2}\pi - 3a + \sin 4a - \tfrac{1}{8}\sin 8a\right)\tan a$$

$$\left. + \left(\pi - 3a - \tfrac{1}{2}\sin 2a + \sin 4a\right)\left(\tfrac{3}{2}a - \sin 2a + \tfrac{1}{8}\sin 4a\right)\operatorname{cosec}^2 a\right\}.$$

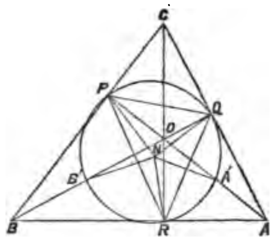
6. Using the expression in Art. 5 for $a = \frac{1}{3}\pi$, that in Art. 4 for $a = \frac{2}{3}\pi$ and $a = \pi$, and either of the two for $a = \frac{1}{2}\pi$, we obtain

$$p\left(\tfrac{1}{3}\pi\right) = 0,\ p\left(\tfrac{1}{2}\pi\right) = \frac{4}{\pi^2} - \frac{1}{8},\ p\left(\tfrac{2}{3}\pi\right) = \frac{27}{8\pi^2} + \frac{\sqrt{3}}{12\pi} + \frac{2}{9},\ p(\pi) = 1.$$

Thus $p\left(\frac{1}{2}\pi\right) = \cdot 2803$ nearly, and $p\left(\frac{2}{3}\pi\right) = \cdot 6101$ nearly.]

1754. (Proposed by T. COTTERILL, M.A.)—If a triangle and its nine-point circle are drawn; in the parabola which touches two sides and the perpendiculars to them through the opposite angles, the focus, the directrix, and the tangents to it from the centre of the circle are given by the figure.

Solution by R. WARREN, B.A.; J. DALE; E. FITZGERALD; *and others.*



Let the parabola touch the sides CA, CB, and the perpendiculars AP, BQ; and let N be the centre of the nine-point circle. Then, since the circle circumscribing the triangle formed by any three tangents to a parabola, passes through the focus, the point R is the *focus* of the parabola. Again, since the locus of the intersection of rectangular tangents to a parabola is the directrix, the line PQ is the directrix of the parabola. Moreover, if the nine-point circle meet OA, OB in A′, B′ respectively, NA′ and NB′ are tangents to the parabola; since, as is easily proved, NA′, NB′ are respectively perpendicular to RQ, RP.

1753. (Proposed by Dr. BOOTH, F.R.S.)—The tangential equation of the surface of the centres of curvature of an ellipsoid being

$$(\xi^2+v^2+\zeta^2)^2 = \left(\frac{\xi^2}{a^2}+\frac{v^2}{b^2}+\frac{\zeta^2}{c^2}\right)(a^2\xi^2+b^2v+c^2\zeta^2-1);$$

show that any two parallel tangent planes being drawn to the surface of centres and to the ellipsoid, the difference of the squares of the coincident perpendiculars let fall upon them from the common centre is always equal to the square of the coinciding semidiameter of the ellipsoid.

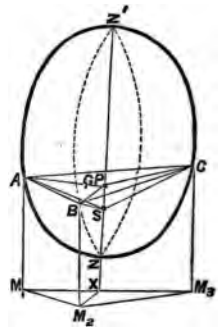
Solution by W. S. BURNSIDE, B.A.

Comparing the equations $\xi x + vy + \zeta z = 1$, and $x\cos\alpha + y\cos\beta + z\cos\gamma = p$ we have $p\xi = \cos\alpha$, $pv = \cos\beta$, $p\zeta = \cos\gamma$. Now, substituting these values of ξ, v, ζ in the given equation of the surface of centres, we have $r^2 = \varpi^2 - p^2$, where $\dfrac{1}{r^2} = \dfrac{\cos^2\alpha}{a^2}+\dfrac{\cos^2\beta}{b^2}+\dfrac{\cos^2\gamma}{c^2}$, and $\varpi^2 = a^2\cos^2\alpha + b^2\sin\beta + c^2\cos^2\gamma$; but ϖ and r are the perpendicular on the tangent plane and the coincident radius vector of the ellipsoid; hence the relation in the Question.

1795. (Proposed by W. K. CLIFFORD.)—(1) Let P be the point in a homogeneous triangular lamina ABC at which the sides subtend equal angles. Show that if the lamina be placed in a smooth prolate spheroid whose long axis is vertical, it will rest in equilibrium when the point P coincides with the lower focus of the spheroid.

(2) If the lamina be not homogeneous, and its centre of gravity be given, construct for the corresponding position of the point P.

Solution by the REV. J. L. KITCHIN, M.A.



1. Let ABC be the triangle in the spheroid, ZZ′ the axis, M_1XM_3 the directrix of its generating ellipse. Let AM_1, BM_2, CM_3 be perpendicular to the plane described by the line M_1X revolving with the generating ellipse; and let e be the eccentricity and S the common focus of the sections. Join AS, BS, CS. Then the height of the centre of gravity (G) of the triangle above the plane $M_1XM_2 = \frac{1}{3}(AM_1 + BM_2 + CM_3) = \frac{1}{3}e^{-1}(AS + BS + CS)$, and this must be a minimum. Now $AS + BS + CS$ is made a minimum, first, by S being a point in the triangle ABC, and secondly, by its coinciding with the point P whose position is determined by subtending equal angles with the sides. Whence the position of equilibrium.

2. In the second case, the height of G above the plane $M_1XM_2 = m\,.\,AM + n\,.\,BM_3 + p\,.\,CM_3 = e^{-1}(m\,.\,AS + n\,.\,BS + p\,.\,CS)$, where m, n, p are constants depending upon the position of G; and this will be a minimum, first by S being at a point (P) in the triangle, and secondly by $m\,.\,AP + n\,.\,BP + p\,.\,CP$ being a minimum. We have then this problem, "to find a point in a triangle such that the sum of its distances from the angular points, multiplied by given constants, may be a minimum;" a problem which has been solved by Fuss in the *Nova Acta Petrop.*, by Bertrand in *Liouville's Journal*, by Wallace in his *Theorems and Formulæ*, p. 120, and by many others. The point P so determined is the point where S lies in the triangle in the position of equilibrium.

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1177. Two rods, connected at their middle points by a string whose length exceeds half the sum of the lengths of the rods, are thrown at random on the ground; what is the chance that they will rest across each other? 32

1263. Dans un triangle quelconque ABC, on mène, à partir des sommets, trois droites dans des directions quelconques; ces trois droites par leurs intersections donnent naissance à un second triangle DEF. Par les sommets de celui-ci on fait passer des droites respectivement parallèles aux côtés du triangle ABC; on obtient ainsi un troisième triangle LMN. Démontrer que la surface du triangle DEF est moyenne proportionnelle entre les surfaces des deux autres. 77

1320. Quid faciam, docti, carum visurus amicum,
Quem late extensâ degere valle juvat?
Hujus ab æde domus tredecim mea millia distat,
Quadrigis rapidis attamen ire queam
Cauponam versus distantem millia bis sex;
Millia cauponâ quinque et amīcus abest.
Quadrigis hora sex millia curritur unâ,
Quatuor interea millia vado pedes.
Quam longe utemur quadrigis, dicite tandem,
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1472. 1. Find two positive rational numbers such that if from each of them, and also from the sum of their squares, their product be subtracted, the three remainders may be rational square numbers.

2. Find two positive rational numbers such that if from each of them, and also from the square root of the sum of their squares, their product be subtracted, the three remainders may be rational square numbers. 60

1480. Prove that if through the middle point of either diagonal of any of the three quadrilateral faces of a tetrahedral frustum, and the middle points of the two edges which meet but are not in the same face with that diagonal, a plane be drawn, the six planes thus obtained will touch the same cone......... 84

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1521. Show that, in a geometric progression of an odd number of terms, the arithmetic mean of the odd-numbered terms is greater than the arithmetic mean of the even-numbered terms, if the common ratio be any positive rational quantity not equal to unity............ 98

1554. Show that, in the ellipse and its circles of maximum and minimum curvature respectively, the semi-ordinates through the focus of the ellipse are

for the circle of maximum curvature... $y_1 = a(1-e)(1+2e)^{\frac{1}{2}}$,

for the ellipse............ $y_2 = a(1-e^2)$,

for the circle of minimum curvature... $y_3 = \dfrac{a\left\{(1-e^2+e^4)^{\frac{1}{2}}-e^2\right\}}{(1-e^2)^{\frac{1}{2}}}$,

and that these values are in the order of increasing magnitude. 99

1583. A system of similar ellipses passes through a fixed point which coincides with one end of their parameters; find the locus of their centres, the envelope of the other parameters, and also that of the system when the locus of the focus is (1) a straight line, (2) a circle 96

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1641. An ellipse is placed with its major axis vertical; find, geometrically, the straight line of quickest descent from the upper focus to the curve. 66

1655. Let the equations of two circles (A) and (B), whose radii are r and r', be $\Theta = 0$ and $\Theta' = 0$; then the two circles (C) and (D), whose equations are $\dfrac{\Theta}{r} - \dfrac{\Theta'}{r'} = 0$ and $\dfrac{\Theta}{r} + \dfrac{\Theta'}{r'} = 0$, intersect at right angles. 95

1682. Find a curve such that the radii of curvature at any two points are to each other as the cubes of the tangents drawn from those points to meet each other. 27

1683. Of all the concurrently-connectant triangles that can be inscribed in a given triangle, prove that the maximum is that for which the concurrent point is the centroid of the given triangle. Find also the maximum triangle when the concurrent point lies on the perpendicular from one angle on the opposite side, or on the bisector of the angle; and find, furthermore, the locus of the concurrent point when the area of the connectant triangle is *constant*. 29

1694. Show that if tangents at right angles are drawn to a cycloid, the locus of their intersection is a trochoid, whose generating circle is equal to that of the cycloid (radius r) and rolls along the line which bisects the height of the cycloid at right angles; the distance of the describing points from the centre

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being $\frac{1}{3}\pi r$. Show how the *loops* of this trochoid form part of the locus. 26

1699. Let O_1, O_2, O_3 be the centres of the escribed circles touching the sides BC, CA, AB, respectively, of the triangle ABC; and P_1, P_2, P_3 the feet of the perpendiculars from the vertices A, B, C on those sides; prove that O_1P_1, O_2P_2, O_3P_3 intersect in a point the sum of the trilinear coordinates of which is

$$\frac{R+r}{R-r}$$ 47

1700. The line DR is perpendicular to the diameter AB of a given semicircle AQB; it is required to find in the circumference a point Q such that, if we join Q with P, a point anyhow given in the line DA, and draw QF perpendicular to DR, the sum or difference of PQ and QF may be given. 94

1703. Given the length of the connecting rod of a horizontal steam engine, and the length of the stroke; find the locus of a given point in the connecting rod during one revolution of the crank. 48

1720. A given curve moves in its own plane, without rotation, so as always to pass through a fixed point A; in any of its positions draw a tangent at A, and a second tangent cutting this at right angles; and find the envelope of the latter. 78

1730. Show that (I) the condition in order that the roots k_1, k_2, k_3 of the equation

$$\gamma k^3 + (-g - \tfrac{1}{2}\alpha + \tfrac{1}{2}\beta + \tfrac{3}{2}\gamma)\,k^2 + (-g - \tfrac{3}{2}\alpha - \tfrac{1}{2}\beta + \tfrac{1}{2}\gamma)\,(k-\alpha) = 0$$

may be connected by a relation of the form $k_3(k_1-k_2)-(k_2-k_3)=0\ldots(1)$; and (II) the result of the elimination of a, b, c from the equations

$$a^2(b+c) = -2\alpha\ldots\ldots(2),\quad b^2(c+a) = 2\beta\ldots\ldots(3),$$
$$c^2(a+b) = -2\gamma\ldots\ldots(4),\quad (b-c)(c-a)(a-b) = -4g\ldots\ldots(5);$$

are each

$$4(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\,g^3 + 4(-\Sigma\alpha^3\beta + 4\Sigma\alpha^2\beta^2 - 2\Sigma\alpha^2\beta\gamma)\,g^2 + (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\,g + 2(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2 = 0$$ 37

1758. If two tangents to a cycloid include a constant angle, show that their sum has a constant ratio to the included arc of the curve. 95

1762. From the intersection of two tangents to a circle to draw a line, cutting the circumference in two points such that, if they are joined with the points of contact of the tangents, the rectangle contained by either pair of the opposite sides of the quadrilateral thus formed may be given or a maximum. 32

1764. The same being supposed as in Quest. 1700; it is required to determine Q, so that if QR be drawn making any given angle with DR, the sum, or difference, of PQ and QR may be given. 94

1778. If the line given by the equation $\alpha x + \beta y + \gamma z = 0$ intersect the conic $(a, b, c, f, g, h)(x, y, z)^2 = 0$ in the points P, Q; the tangents to the conic at these points meeting in the point R, and a focus being at the point F; prove that

$$\frac{FP\,.\,FQ}{FR^2} = \frac{\Pi^2}{\Pi^2 - \Theta\Sigma},$$

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where $\Theta = (A, B, C, F, G, H)(\sin A, \sin B, \sin C)^2,$

$\Sigma = (A, B, C, F, G, H)(\alpha, \beta, \gamma)^2,$

$2\Pi = \sin A \frac{d\Sigma}{d\alpha} + \sin B \frac{d\Sigma}{d\beta} + \sin C \frac{d\Sigma}{d\gamma}$ 27

1782. According as one side of a triangle is a geometric, harmonic, or arithmetic mean between the other two, so is the cosine of one of the semi-angles of the *pedal triangle* a geometric, harmonic, or arithmetic mean between the cosines of the semi-angles of the other two. 31

1783. Eliminate θ between each of the following sets of equations:

$$\left.\begin{aligned} X\cos(\theta-A)+Y\cos\theta &= 2R\sin(\theta+C)\cos\theta\cos(\theta-A) \\ X\sin(\theta-A)+Y\sin\theta &= -2R\cos(\theta+C)\sin\theta\sin(\theta-A) \end{aligned}\right\};$$

$$\left.\begin{aligned} Y-X\cot(B+\tfrac{1}{2}\theta) &= R\left\{\cos C-\sin(C-\theta)\cot(B+\tfrac{1}{2}\theta)\right\} \\ Y+X\tan(B+\tfrac{1}{2}\theta) &= R\left\{\cos C-\sin(C-\theta)\tan(B+\tfrac{1}{2}\theta)\right\} \end{aligned}\right\};$$

where A, B, C are the angles of a triangle, and R the radius of its circumscribing circle. 80

1791. Given a quartic curve $U = 0$, to find three cubic curves $P = 0$, $Q = 0$, $R = 0$, each meeting the quartic in the same six points 1, 2, 3, 4, 5, 6, and such that $P = 0$, $R = 0$ may besides meet the quartic in the same three points a, b, c, and that $Q = 0$, $R = 0$ may besides meet the quartic in the points α, β, γ. 17

1792. Find the condition in order that the normals to the conic $(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}}$, drawn at the points of contact of the sides of the triangle of reference, may meet in a point......... 18

1798. (1.) Let $f(x) = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} \dots + \frac{x^n}{1.2\dots n}$;

prove that $f(x)$ cannot have two real roots.

(2.) Let $\phi(x) = 1 + \nu x + \frac{\nu(\nu+1)}{1.2}x^2 + \dots + \frac{(\nu+1)\dots(\nu+n-1)}{1.2\dots n}x^n$,

if $\nu > 0$ or $< -n$, prove that $\phi(x)$ cannot have two real roots.

(3.) Deduce (1) from (2). 67

1801. A thin bar of wood AB, with a groove running along the middle, has a perpendicular arm CD projecting each way; and a similar piece EGAF is connected with it by a pivot at A. In EGAF there is a small sharp-edged wheel at G, turning on a pivot which lies in the plane of EGAF, and in the direction of EGF. A pencil is inserted in the concourse of both grooves at H; and CAD being slid along the axis of X, two curves are generated. Find them, and show that the curve described by the pencil is the evolute of that described by the wheel.. 93

1810. Prove that the value of the expression

$$\left\{\cos(\alpha-\beta)-\cos(\gamma-\delta)\right\}^2 + \left\{\cos(\alpha+\beta)-\cos(\gamma+\delta)\right\}^2 + \left\{\cos(\alpha+\gamma)\cos(\beta-\delta)-\cos(\beta+\delta)\cos(\alpha-\gamma)\right\}^2$$

is unaltered by the interchange of β, γ. 24

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1813. A luminous surface of uniform intrinsic brightness B is of the form generated by the revolution of a catenary about its axis; prove that the illumination of a small plane area δ situated at the intersection of the axis and directrix of the catenary, and having its plane perpendicular to the axis, is $\dfrac{4\pi B\delta}{(e^{\alpha}+e^{-\alpha})^2}$, α being such that $e^{2\alpha} = \dfrac{\alpha+1}{\alpha-1}$ 18

1815. Show that the feet of the perpendiculars drawn from the two points (l, m, n), (l^{-1}, m^{-1}, n^{-1}), upon the sides of their triangle of reference, all lie on the same circle; and find its equation. 19

1817. In lines of the third order, prove that the locus of the middle points of chords parallel to an asymptote which does not *cut the curve*, is a straight line; but when the asymptote cuts the curve, show that the locus then becomes a hyperbola...... 70

1818. Two points being taken at random within (1) a circle, or (2) a sphere, find the probability that the chord drawn through them is less than a given line. 20

1820. Prove that

$$\frac{m^m}{x+m} - \frac{m}{1} \cdot \frac{(m-1)^m}{x+m-1} + \frac{m(m-1)}{1.2} \frac{(m-2)^m}{x+m-2} - \&c.$$
$$= \frac{1.2\ldots.m.x^{m-1}}{(x+1)(x+2)\ldots(x+m)} \text{ .... } 100$$

1822. Prove that the area of a triangle circumscribing a conic is $ab^{-2}p_1p_2p_3$, where p_1 is one of the four perpendiculars from the vertex (1) on the focal vectors to the points of contact of tangents from the same vertex....................................... 24

1824. Assuming that the bowler can run with any velocity less than v, and the batter can hit with any velocity less than u, and all less velocities and all directions (along the ground only) are equally probable; find the chance that the bowler will be able to stop a hit of the batter.. 21

1825. Prove that, in space, the locus of a point such that, if perpendiculars be drawn from it to the faces of a tetrahedron, their feet shall lie in a plane, is the surface

$$\frac{A}{x} + \frac{B}{y} + \frac{C}{z} + \frac{D}{w} = 0,$$

A, B, C, D representing the areas of the faces, and x, y, z, w the perpendiculars drawn on them from any point. 45

1826. The vertex of a triangle is fixed, while its base, of constant length, moves along a given line; show that the locus of the centre of the circumscribed circle is a parabola. 28

1827. Find the values of x, y, z which make the function $u = nf(x) \cdot \phi(y) \cdot \psi(z)$ a maximum; x, y, z being connected by the equation $a^{f(x)-\alpha} \cdot b^{\phi(y)-\beta} \cdot c^{\psi(z)-\gamma} = A$. 29

1834. 1. It is required to find on a given cubic curve three points A, B, C, such that, writing $x = 0$, $y = 0$, $z = 0$ for the

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equations of the lines BC, CA, AB respectively, the cubic curve may be transformable into itself by the inverse substitution $(\alpha x^{-1}, \beta y^{-1}, \gamma z^{-1})$ in place of x, y, z respectively, α, β, γ being disposable constants.

2. In the cubic curve $ax(y^2+z^2)+by(z^2+x^2)+cz(x^2+y^2)+2lxyz=0$ the inverse points (x, y, z) and (x^{-1}, y^{-1}, z^{-1}) are corresponding points (that is, the tangents at these two points meet on the curve) 38

1835. Three lines being drawn at random on a plane, determine the probability that they will form an acute triangle. 70

1836. Through the extremities of a diameter of an hyperbola (or its conjugate) at right angles to one asymptote, straight lines are drawn parallel to the other; if the straight lines joining the extremities of the diameter to any point on the curve be produced, they will intercept on the parallels portions whose difference is constant. 100

1837. P is any point in the plane of a circle (C); Q any point on the polar of P with respect to (C); show that (C) cuts orthogonally the circle on PQ as diameter. 22

1838. Two steamers are continually running between a port and two given points, subtending a given angle at the port, and each of which is just visible from it; find the chance of the steamers being visible to one another at any particular instant. 53

1840. If, when L, M, N are three collinear points, [LMN] denote $+1$ or -1 according as M is within or external to the segment LN; prove the following theorems of four collinear points A, B, C, D, anyhow situated relatively to one another:—

(1)...[ACB] [CAB] = [BDC] [DBA] = $-$[ADC] [DAB] = $-$[BCD] [CBA];

(2)...$AC^2+BD^2-AD^2-BC^2=2$ [ACD] [CAB] AB . CD;

(3)...[ACD] [CAB] AB . CD + [ADB] [DAC] AC . DB + [ABC] [BAD] AD . BC = 0. 21

1842. Find the condition connecting the coefficients of two binary quartics, in order that there may be an arrangement of their roots $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ which will make

$$\begin{vmatrix} \alpha\alpha', & \alpha, & \alpha', & 1 \\ \beta\beta', & \beta, & \beta', & 1 \\ \gamma\gamma', & \gamma, & \gamma', & 1 \\ \delta\delta', & \delta, & \delta', & 1 \end{vmatrix} = 0.$$ 23

1848. Supposing the density of the population of the metropolitan area (radius 8 miles) to vary inversely as the distance from the centre, find the probability of two persons taken at random living nearer than 8 miles to each other........ 101

1851. Given four points in a plane; show that the equation which determines the coefficient of xy, in any conic passing through the four points, so that the circumscribing rectangle may be a maximum or a minimum, is of the third order. 64

1854. Solve the differential equation $\frac{dy}{dx}+by^2=ax^2$; or, differential

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added to multiple of square of dependent variable equal to multiple of square of independent variable........................ 50

1857. If for shortness we put

$P = x^3 + y^3 + z^3,\ Q = yz^2 + y^2z + zx^2 + z^2x + xy^2 + x^2y,\ R = xyz,$

$P_0 = a^3 + b^3 + c^3,\ Q_0 = bc^2 + b^2c + ca^2 + c^2a + ab^2 + a^2b,\ R_0 = abc;$

then (α, β, γ) being arbitrary, show that the cubic curves $\begin{vmatrix} \alpha, & \beta, & \gamma \\ P, & Q, & R \\ P_0, & Q_0, & R_0 \end{vmatrix} = 0$ pass all of them through the same nine points, lying six of them upon a conic and three of them upon a line; and find the equations of the conic and line, and the coordinates of the nine points of intersection; find also the values of $(\alpha : \beta : \gamma)$ in order that the cubic curve may break up into the conic and line. 37

1858. If in any symmetric function of the differences of the roots of an equation, each root a_k be changed into $\frac{1}{(a_k - x)}$, show that the result, when cleared of fractions, will be a covariant...... 54

1859. Show that the locus of the centres of equilateral hyperbolas touching the sides of a given obtuse-angled triangle is the self-conjugate circle of this triangle................................ 36

1861. Prove that the difference between the sum of the sines and the sum of the cosines is greater or less than unity according as the triangle is acute or obtuse-angled. 42

1864. Prove that

$$(1)\ldots 1 - n + \frac{n(n-1)}{1.2} - \ldots \pm \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r} = \pm \frac{(n-1)(n-2)\ldots(n-r)}{1.2\ldots r},$$

$$(2)\ldots \frac{1}{m+1} + \frac{1}{m+2} \ldots + \frac{1}{m+n} = \frac{n(n+1)\ldots(n+m)}{1.2\ldots m} \left\{ \frac{1}{(m+1)^2} - \frac{n-1}{1} \cdot \frac{1}{(m+2)^2} + \&c. \right\}$$ · 23

1867. From a point O_1 on a conic, $(n-1)$ lines are drawn to points $O_2, O_3 \ldots O_n$ on the conic; $O_1O_3, O_1O_4 \ldots O_1O_n$ being inclined at angles $\alpha_3, \alpha_4, \alpha_n$ to O_1O_2; find the product of $O_1O_2 . O_1O_3 \ldots O_1O_n$, (1) in the general case, (2) when the conic becomes a circle, (3) when $O_1O_2O_3 \ldots O_n$ is a regular polygon in the circle 46

1868. Three straight lines are drawn at random on an infinite plane, and a fourth line is drawn at random to intersect them; find the probability of its passing through the triangle formed by the other three 82

1871. The envelope of a circle whose diameter is a chord, fixed in direction, of a given conic, is another conic whose foci are at the extremities of that diameter of the former which is conjugate to the fixed direction. Prove this, and find where the circle touches its envelope. 40

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1873. Assuming that all lives are of equal duration, what must that duration be, in order that the births, deaths, and consequent increase or decrease of population, may remain unchanged? 71

1876. If three of the roots of the equation $(a, b, c, d, e)\ (x, 1)^4 = 0$ be in arithmetical progression, show that $55296H^3J - 2304aH^2I^2 - 16632a^2HIJ + 625a^3I^3 - 9261a^3J^2 = 0$, where $H = ac - b^2$, $I = ae - 4bd + 3c^2$, $J = ace + 2bcd - ad^2 - b^2e - c^3$...... 58

1877. Let P be the intersection of the three perpendiculars; O the centre of the circumscribed circle (radius = R); α, β, γ the middle points of the sides, of any triangle ABC. On the segments PA, PB, PC let the three points p, q, r be taken, such that $Pp = \frac{1}{n} . PA$, $Pq = \frac{1}{n} . PB$, $PR = \frac{1}{n} . PC$; and on $P\alpha$, $P\beta$, $P\gamma$ three other points p', q', r', such that $Pp' = \frac{2}{n} . P\alpha$, $Pq' = \frac{2}{n} . P\beta$, $Pr' = \frac{2}{n} . P\gamma$. Prove (1) that the lines pp', rr' intersect on the line PO in a point M, such that $PM = \frac{1}{n} . PO$; (2) that the six points in question lie on a circle whose centre coincides with M, and whose radius $= \frac{1}{n} . R$; (3) that this circle will touch the circle inscribed in the triangle, if $\frac{1}{n} = \frac{1}{2}$ or $= 1 + \frac{r^2}{\rho^2}$, where r, ρ are the radii of the inscribed and self-conjugate circles of the triangle......... 71

1879. If forces represented by the sides of a plane hexagon taken in order are in equilibrium, the directions of the sides of the two triangles formed by joining alternate points of the hexagon are in involution. 73

1883. Draw a straight line parallel to a given straight line to cut a given semicircle so that the trapezoid formed by the chord, the diameter, and the perpendiculars on the diameter from the points of section may be given or a maximum. 54

1885. Investigate the following constructions for determining the point (T) of intersection of the common tangents of an ellipse and its circle of curvature at P. If O be the centre of the circle, C that of the ellipse, S either focus; then (1) T lies on the confocal hyperbola which passes through P; (2) OC bisects PT; and (3) SP, ST are equally inclined to OS. 87

1887. Find the mean value of the volume of a tetrahedron, three of whose vertices lie respectively in three non-intersecting edges, and the fourth at the centre of a given parallelepiped.......... 35

1888. (1.) Amongst the conics which have three-pointic contact with a cubic at a given point, there are, in general, three which have a three-pointic contact elsewhere and a fourth passes through the points of contact of these three with the

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cubic. The number of such conics is reduced to one, when the cubic has a cusp.

(2.) Amongst the conics which have four-pointic contact with a cubic at a given point there are three which touch the cubic elsewhere. There is but one such conic when the cubic has a node, and none when it has a cusp. 55

1890. Find the equation of a conic passing through three given points and having double contact with a given conic 88

1893. If the edges of any hexahedron meet four by four in three points, then the four diagonals meet in a point.................. 89

1894. Supposing n chords to be drawn at random in a given circle, determine the several probabilities that there shall be $0, 1, 2, 3, \ldots\ldots \frac{1}{2}n(n-1)$ intersections 110

1895. Two circles A and B, whose radii are a and b, touch at two points P and Q a common circle whose radius is r; show that the length of their common tangent (AB), external or internal according as their contacts with it are of similar or opposite species, is given by the formula $(AB) = \frac{\sqrt{(r \pm a) \cdot (r \pm b)}}{r} \cdot (PQ)$;

and hence prove immediately the following extension of Ptolemy's Theorem given by Mr. Casey. When four circles A, B, C, D touch a common circle, the six common tangents AB, &c., of their six groups of two external or two internal according as the contacts of the two with the common circle are similar or opposite, are connected by the relation

$(BC) \cdot (AD) + (CA) \cdot (BD) + (AB) \cdot (CD) = 0$ 90

1901. Find the curve whose circle of curvature always passes through a fixed point 91

1909. If λ and λ' be the angles which any two conjugate diameters AB and CD of an ellipse subtend at any point P in the curve, and a the angle which either axis subtends at an extremity of the other axis; prove that $\cot^2\lambda + \cot^2\lambda' = \cot^2 a$. 73

1915. If $\left.\begin{array}{l} C_r = \frac{1}{x}\cos\frac{r\pi}{2m} + \frac{1}{x^2}\cos\frac{2r\pi}{2m} + \frac{1}{x^3}\cos\frac{3r\pi}{2m} + \&c. \\ \text{and } S_r = \frac{1}{x}\sin\frac{r\pi}{2m} + \frac{1}{x}\sin\frac{2r\pi}{2m} + \frac{1}{x^5}\sin\frac{3r\pi}{2m} + \&c. \end{array}\right\}$ to ∞, show that

$(P_1) = (C_1^2 + S_1^2)(C_3^2 + S_3^2) \ldots (C_{2m-1}^2 + S_{2m-1}^2) = (x^{2m}+1)^{-1}$,

$(P_2) = (C_0^2 + S_0^2)^{\frac{1}{2}} C_{2m}^2 + S_{2m}^2)^{\frac{1}{2}}(C_2^2 + S_2^2) .. (C_{2m-2}^2 + S_{2m-2}^2) = (x^{2m}-1)^{-1}$ 110

1922. Let AA_1, BB_1 be the major and minor axes of an ellipse, and CP, CD any pair of semi-conjugate diameters; draw AG, BH, B_1H_1 perpendicular to CP, and A_1g, B_1h, Bh_1 perpendicular to CD; also let AG, A_1g meet in Q_1; BH, B_1h in Q_2; AG, Bh_1 in R_1; A_1g, BH in R_2; A_1g, B_1H_1 in R_3; and AG, B_1h in R_4. Prove that the sum of the areas of the loci of Q_1, Q_2 is equal to the sum of the areas of the loci of R_1, R_2, R_3, R_4 107

1925. Given four points on a circle whose radius is r; show that the centroids (centres of gravity of the areas) of the four triangles that can be formed from them lie on another circle, whose radius is $\frac{1}{3}r$. 92

No.
1950. If A, B, C, D be four points in a circle; and if AB, CD produced meet in F, and AD, BC produced meet in G, prove that the lines which bisect the angles F and G are at right angles to each other 105

1957. Show that the chords of quickest and slowest descent from the highest point of an ellipse in a vertical plane are at right angles to each other and parallel to the axes of the curve... 102

1965. Four conics through four points form a harmonic system; prove that if two conjugates be a circle and an equilateral hyperbola, the other two must be of equal eccentricities...... 103

1968. If from any point P in a circle concentric with a given ellipse, and the radius of which is equal to the distance between the ends of the major and minor axes, a pair of tangents be drawn to the ellipse and produced to meet the circle in the points S and S′, prove that the line SS′ is parallel to the polar of P... 104

Unsolved Questions.

1448. Proposed by W. K. Clifford, Trinity College, Cambridge.)—The sides of a triangle repel with a force varying inversely as the cube of the distance; find the position in which a particle will rest.

Also, supposing the faces of a tetrahedron to repel according to the same law, find where a particle will rest.

1524. (Proposed by the Rev. T. P. Kirkman, M.A., F.R.S.)—Let N be any odd number, and e_1, e_2, e_3...different positive numbers; also let

$$\mathrm{N}-\mathrm{N}' = 2m_1e_1+2m_2e_2+2m_3e_3+\&c. \quad \ldots\ldots\ldots\ldots\ldots(\mathrm{A})$$

be in turn every partition of that form of every even number $\overline{\overline{>}}\ 0$ and $<$ N, ($\mathrm{N}-\mathrm{N}' = 2.0.e_1$ being the case of $\mathrm{N}'=\mathrm{N}$); then

$$\frac{\mathrm{N}(\mathrm{N}-2)(\mathrm{N}-4)\ldots\ldots3.1}{(\mathrm{N}-1)(\mathrm{N}-3)\ldots\ldots4.2} = \Sigma\frac{1}{(2e_1)^{m_1}.\Pi m_1.(2e_2)^{m_2}.\Pi m_2\ldots}$$

(Πm being $1.2\ldots m$, and $\Pi 0 = 1$), where every partition (A) gives a term of the sum Σ, and all partitions are different which have not the same e_1, e_2, e_3...(thus $9-1 = 2.4.1 = 2.1.4 = 2.2.1+2.1.2$ are all different partitions.)

1591. (Proposed by Professor Hirst.)—Find the polar equation of a curve whose radii vectores are each divided into segments having a *constant* ratio when, upon the same, the respective centres of curvature are projected orthogonally.

1787. (Proposed by Professor Cremona.)—On donne une conique K et un point p. Une transversale menée arbitrairement par p rencontre K en deux points m, m'; et soit x un point de la transversale tel que le rapport anharmonique ($pxmm'$) soit un nombre λ donné. Trouver le lieu du point x. Si λ est l'une des racines cubiques imaginaires de -1, on a une certaine conique C (p). De quelle manière change C (p), si l'on fait varier p?

Recherche analogue par rapport à une surface du second ordre.

1792. (Proposed by R. Warren, B.A.)—Find the condition in order that the normals to the conic $(l\alpha)^{\frac{1}{2}}+(m\beta)^{\frac{1}{2}}+(n\gamma)^{\frac{1}{2}}$, drawn at the points of contact of the sides of the triangle of reference, may meet in a point.

Solution by the Editor.

The normal perpendicular to the side γ passes through the point $(\gamma,\ l\alpha-m\beta)$, hence, writing (λ, μ, ν) for $(\cos A, \cos B, \cos C)$, the trilinear equation of this normal is readily found to be $l\alpha-m\beta+(l\mu-m\lambda)\gamma=0$.

The equations of the other two normals are of course similar to this; and the condition of the concurrence of the three is

$$\begin{vmatrix} l, & -m, & l\mu-m\lambda \\ m\nu-n\mu, & +m, & -n \\ -l, & n\lambda-l\nu, & +n \end{vmatrix}=0,$$

which, when expanded, may be expressed in the form

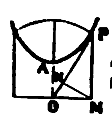
$$a^2mn(m\nu-n\mu)+b^2nl(n\lambda-l\nu)+c^2lm(l\mu-m\lambda)=0.$$

The locus of the centres of the conics which satisfy this condition of concurrence is a cubic possessing many interesting properties: for which see Quest. 1545 (*Reprint*, Vol. II., p. 57), also a paper in the *Messenger of Mathematics*, Vol. III., p. 15.

1813. (Proposed by the late H. J. Purkiss, B.A.)—A luminous surface of uniform intrinsic brightness B is of the form generated by the revolution of a catenary about its axis; prove that the illumination of a small plane area δ situated at the intersection of the axis and directrix of the catenary, and having its plane perpendicular to the axis, is

$$\frac{4\pi B\delta}{(e^{a}+e^{-a})^2},\ a \text{ being such that } e^{2a}=\frac{a+1}{a-1}.$$

Solution by the Proposer; H. M. Taylor, B.A.; J. Dale; *the* Rev. J. L. Kitchin, M.A.; *and others.*



Let OA be the axis, OM the directrix of the generating catenary. Let OP be drawn to touch the curve at P. Draw the ordinate PM, and MN perpendicular to OP. Let OM $=x$, PM $=y$, OA $=$ MN $=c$, $\frac{x}{c}=a$. Then the illumination at O is the same as would be produced by the portion of a spherical surface (of the same intrinsic brightness), having O for its centre and OP for its radius, bounded by the conical surface formed by the revolution of OP about OA. Now the projection of this portion of surface on the plane to be illuminated is a circle of radius OM. Hence the expression for the illumination will be

$$\frac{\pi\,.\,OM^2\,.\,B\,.\,\delta}{OP^2}=\frac{\pi B\delta\,.\,MN^2}{MP^2}=\pi B\delta\left(\frac{c}{y}\right)^2=\frac{4\pi P\delta}{(e^{a}+e^{-a})^2}.$$

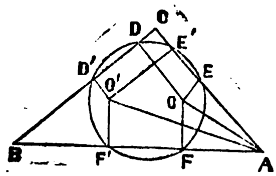
To determine a, we have clearly at P

$$\frac{y}{x}=\frac{dy}{dx}=\frac{e^a-e^{-a}}{2}; \therefore \frac{e^a+e^{-a}}{e^a-e^{-a}}=a; \therefore e^{2a}=\frac{a+1}{a-1}.$$

This equation has only one real positive root, which is greater than unity. For e^{2a} is always positive, whereas $\frac{a+1}{a-1}\left(=1+\frac{2}{a-1}\right)$ is only positive when $a>1$, and we see moreover that it diminishes as a increases. These results are of course obvious from the geometry of the figure.

1815. (Proposed by J. GRIFFITHS, M.A.)—Show that the feet of the perpendiculars drawn from the two points (l, m, n), (l^{-1}, m^{-1}, n^{-1}), upon the sides of their triangle of reference, all lie on the same circle; and find its equation.

Solution by H. M. TAYLOR, B.A.



Let any circle cut the sides BC, CA, AB of a triangle in D, D'; E, E'; F, F'; from E, E' and F, F' draw perpendiculars to CA, AB meeting in O, O'; and join AO, AO'. Let $\angle$OAE $=\theta$, OAF $=\phi$, O'AE' $=\theta'$, O'AF' $=\phi'$; then $\theta+\phi=A=\theta'+\phi'$, or $\theta'-\theta=\phi-\phi'$, whence $\cos\theta'\cos\theta+\sin\theta'\sin\theta=\cos\phi\cos\phi'+\sin\phi\sin\phi'$. But by properties of the circle, AE . AE' = AF . AF', whence $\cos\theta\cos\theta'=\cos\phi\cos\phi'$; therefore $\sin\theta\sin\theta'=\sin\phi\sin\phi'$; but $\sin\theta:\sin\phi=m:n$, and $\sin\theta':\sin\phi'=m':n'$; therefore $mm'=nn'=ll'$, by symmetry.

This proves that a circle passes through the six points which are the feet of the perpendiculars from (l, m, n), (l^{-1}, m^{-1}, n^{-1}).

Again, let $(\alpha_1, \beta_1, 0)$, $(\alpha_2, \beta_2, 0)$ be the coordinates of F, F';

then $\qquad \alpha_1=\text{OD}+\text{OF}\sin\text{B}$, and $\beta_1=\text{OE}+\text{OF}\sin\text{A}$,

therefore $\qquad \dfrac{\alpha_1}{\beta_1}=\dfrac{l+n\sin\text{B}}{m+n\sin\text{A}}$, and similarly $\dfrac{\alpha_2}{\beta_2}=\dfrac{l^{-1}+n^{-1}\sin\text{B}}{m^{-1}+n^{-1}\sin\text{A}}$.

The quadratic which has these roots is

$$(m+n\sin\text{A})(m^{-1}+n^{-1}\sin\text{A})\,\alpha^2+(l+n\sin\text{B})(l^{-1}+n^{-1}\sin\text{B})\,\beta^2$$
$$-\left\{(l+n\sin\text{B})(m^{-1}+n^{-1}\sin\text{A})+(l^{-1}+n^{-1}\sin\text{B})(m+n\sin\text{A})\right\}\alpha\beta=0.$$

To find the equation to the circle required, we have only to make this equation homogeneous by adding terms containing γ.

[The theorem follows at once from the consideration that (l, m, n), (l^{-1}, m^{-1}, n^{-1}) are the foci of a conic inscribed in the triangle, and the circle in question is that drawn on the major axis as diameter, since this circle is the locus of the feet of the perpendiculars from the foci on the tangents.]

1818. (Proposed by the EDITOR.)—Two points being taken at random within (1) a circle, or (2) a sphere, find the probability that the chord drawn through them is less than a given line.

Solution by PROFESSOR SYLVESTER.

1. Let 1 be the radius of the given circle, $2k$ the length of the given line, $k'^2 = 1-k^2$, and p the probability that the chord drawn through two arbitrary points within the circle does not exceed $2k$.

Divide the circle into concentric rings each of varying radius r. Then the arrangements out of which are to be sought the favourable cases, become a reduplication of rings of density $2\pi r dr$, combined respectively with circles of radius r. Let 2λ be the length of a chord in any such circle, which produced both ways to meet the *given* circumference generates a chord $2k$; also let A be the area of the smaller segment of the circle to radius r cut off by the chord λ. Then $\lambda^2 = r^2-k'^2$; and so long as $r<k'$ every produced chord *exceeds* $2k$; but when $r>k'$, we have

$$A = \int_0^{\lambda} \frac{2\rho^2 d\rho}{(r^2-\rho^2)^{\frac{1}{2}}}, \quad \pi^2 p = \int_{k'}^{1} 4\pi r dr\,(2A); \text{ or if } \rho = rt,\ 1-\frac{k'^2}{r^2} = u^2,$$

$$\pi p = 4\int_{k'}^{1} 4r^3 dr \int_0^{u} \frac{t^2 dt}{(1-t^2)^{\frac{1}{2}}} = 4k'^4 \int_0^{k} \delta_k \left(\frac{1}{1-k^2}\right)^2 dk \int_0^{k} \frac{k^2 dk}{(1-k^2)^{\frac{1}{2}}}$$

$$= 2\,(\sin^{-1}k - kk') - 4k'^4 \int_0^{k} \frac{k^2 dk}{(1-k^2)^{\frac{5}{2}}} = 2\sin^{-1}k - 2kk' - \tfrac{4}{3}k^3k'.$$

Thus, for example, the chance of the chord not exceeding the ***radius*** is $\frac{1}{3} - \frac{7\sqrt{3}}{12\pi}$, that is ·0117, or a little less than $\frac{1}{85}$.

2. The result is striking for simplicity when for a circular we substitute a spherical contour; for then we have

$$\left(\frac{4}{3}\pi\right)^2 p = 2\int_{k'}^{1} 4\pi r^2 dr \left(\frac{4\pi\lambda^4}{3r}\right) = \frac{32}{3}\pi^2 \int_{k'}^{1} r dr\,(r^2-k'^2)^{\frac{3}{2}} = \frac{16}{9}\pi^2 k',$$

therefore in this case $p = k^6$.

Thus, for example, it is rather more than an even chance that a chord drawn through two points in the interior of a sphere shall *exceed eight-ninths* of the diameter, and exactly 63 against 1 that it shall exceed the *radius*.

[If we consider every possible position of the second point, whether nearer to or farther from the centre than the first, it is easy to see that the chord will not exceed $2k$ if the second point fall within an area S, consisting of two mixtilinear triangles bounded by the circumference of the given circle and two chords, each equal to $2k$, drawn through the first point; thus we find

$$S = 2\left(\cos^{-1}\frac{k'}{r} - k'\lambda\right);$$

therefore $\pi^2 p = \int_{k'}^{1} 2\pi r dr\,(S) = 2\pi \left(r^2 \cos^{-1}\frac{k'}{r} - k'\lambda - \frac{2}{3}k'\lambda^3\right)_{r=k}^{r=1};$

whence we obtain $p = \frac{1}{\pi}(2\sin^{-1}k - 2kk' - \frac{4}{3}k^3k')$, as found in Professor SYLVESTER's Solution.

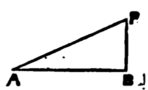
For the sphere we have only to suppose the circle with its connected lines to revolve round a diameter through the first point, then S generates a volume V within which the second point must lie in order that the chord through the two points may not exceed $2k$; and we find

$$V = \tfrac{4}{3}\pi\frac{k^3\lambda}{r}; \therefore (\tfrac{4}{3}\pi)^2 p = \int_{k'}^{1} 4\pi r^2 dr\,(V) = \left(\tfrac{16}{9}\pi^2 k^3\lambda^3\right)_{r=k'}^{r=1};$$

whence we obtain $p = k^6$ as found in Professor SYLVESTER's Solution.]

1824. (Proposed by J. M. WILSON, M.A.)—Assuming that the bowler can run with any velocity less than v, and the batter can hit with any velocity less than u, and all less velocities and all directions (along the ground only) are equally probable, find the chance that the bowler will be able to stop a hit of the batter.

Solution by the PROPOSER.



Let the ball be struck at A with a velocity x in the direction AP, and let the bowler meet it at P by running in the direction BP, making an angle ϕ with AB, with a velocity y. Then $y = \frac{\sin\theta}{\sin\phi}x$, and therefore y is least when ϕ is a right angle for given values of x and θ, and $\sin\theta = \frac{y}{x}$. Hence the bowler can stop all balls (1) whose velocity is less than v, whether before or behind the wicket; (2) whose angle of direction θ is less than $\sin^{-1}\frac{v}{x}$.

The probability required is, therefore,

$$\frac{v}{u} + \int_{v}^{u} \frac{1}{\pi}\sin^{-1}\frac{v}{x}\cdot\frac{dx}{u} = \frac{v}{2u} + \frac{1}{\pi}\left\{\sin^{-1}\frac{v}{u} + \frac{v}{u}\log_e\frac{u+\sqrt{(u^2-v^2)}}{v}\right\}.$$

1840. (Proposed by Professor SYLVESTER.)—If, when L, M, N are three collinear points, [LMN] denote $+1$ or -1 according as M is within or external to the segment LN; prove the following theorems of four collinear points A, B, C, D, anyhow situated relatively to one another:—

(1).. [ACD][CAB] = [BDC][DBA] = −[ADC][DAB] = −[BCD][CBA];

(2).. $AC^2 + BD^2 - AD^2 - BC^2 = 2$ [ACD] [CAB] AB . CD;

(3).. [ACD] [CAB] AB . CD + [ADB] [DAC] AC . DB + [ABC] [BAD] AD . BC = 0.

Solution by W. H. LAVERTY.

The sign of [LMN] is the same as that of $\left[\frac{LM}{MN}\right]$, for the latter is positive or negative according as M is within or external to the segment LN; hence, substituting this last notation, we have

$$(\alpha)\quad [ACD]\,[CAB] = \left[\frac{AC}{CD}\right]\left[-\frac{AC}{AB}\right] = \left[-\frac{1}{AB\,.\,CD}\right];\ \text{since } AC^2 \text{ is positive.}$$

$$(\beta)\quad [BDC]\,[DBA] = \left[-\frac{BD}{CD}\right]\left[\frac{BD}{AB}\right] = \left[-\frac{1}{AB\,.\,CD}\right];$$

$$(\gamma)\quad -[ADC]\,[DAB] = -\left[-\frac{AD}{CD}\right]\left[-\frac{AD}{AB}\right] = \left[-\frac{1}{AB\,.\,CD}\right];$$

$$(\delta)\quad -[BCD]\,[CBA] = -\left[\frac{BC}{CD}\right]\left[\frac{BC}{AB}\right] = \left[-\frac{1}{AB\,.\,CD}\right];$$

$$(\epsilon)\quad [ADB]\,[DAC] = \left[\frac{AD}{DB}\right]\left[-\frac{AD}{AC}\right] = \left[-\frac{1}{AC\,.\,DB}\right];$$

$$(\zeta)\quad [ABC]\,[BAD] = \left[\frac{AB}{BC}\right]\left[-\frac{AB}{AD}\right] = \left[-\frac{1}{AD\,.\,BC}\right];$$

hence we see that (α), (β), (γ), (δ), are all equal; which proves (1).

Again, $AC^2 + BD^2 - AD^2 - BC^2 = -2\,.\,AB\,.\,CD$, as we should find by substituting (AB + BC) for AC, &c.; also

$$2\,[ACD]\,[CAB]\,AB\,.\,CD = 2\left[-\frac{1}{AB\,.\,CD}\right] \times AB\,.\,CD = -2\,.\,AB\,.\,CD;$$

which proves (2).

Lastly, substituting from (α), (ϵ), (ζ), in (3), we have

$$-AB\,.\,CD - AC\,.\,DB - AD\,.\,BC = -AB\,.\,CD + (AB + BC)(BC + CD) - (AB + BC + CD)\,BC = 0;$$ which proves (3).

1837. (Proposed by J. GRIFFITHS, M.A.)—P is any point in the plane of a circle (C); Q any point on the polar of P with respect to (C); show that (C) cuts orthogonally the circle on PQ as diameter.

Solution by ARCHER STANLEY; R. WARREN, B.A.; W. H. LAVERTY; REV. J. L. KITCHIN, M.A.; *and others.*

The line joining the centre of the given circle to P cuts the polar of the latter in a point P_1 which is inverse to P, relative to (C). Again, PP_1Q being a right angle, the circle on PQ as diameter passes through the inverse points P and P_1, and consequently cuts (C) orthogonally. (Townsend's *Modern Geometry*, Vol. I., Art. 156.)

1842. (Proposed by R. BALL, M.A.)—Find the condition connecting the coefficients of two binary quartics, in order that there may be an arrangement of their roots $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ which will make.

$$\begin{vmatrix} \alpha\alpha', & \alpha, & \alpha', & 1 \\ \beta\beta', & \beta, & \beta', & 1 \\ \gamma\gamma', & \gamma, & \gamma', & 1 \\ \delta\delta', & \delta, & \delta', & 1 \end{vmatrix} = 0.$$

Solution by the PROPOSER; R. WARREN, B.A.; *and others.*

Let $(a, b, c, d, e)(x, 1)^4 = 0$; put $x = \dfrac{l+my}{\lambda+\mu y}$, and suppose for y we have $(a', b', c', d', e')(y, 1)^4 = 0$; then $I' = M^4 I$, $J' = M^6 J$, where I, J are the invariants of the quartics, and M the modulus of transformation. Let $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$ be the corresponding roots of the two equations; then from the relations $\alpha = \dfrac{l+m\alpha'}{\lambda+\mu\alpha'}$, &c., we obtain

$$\mu\alpha\alpha' + \lambda\alpha - m\alpha' - l = 0, \quad \mu\beta\beta' + \lambda\beta - m\beta' - l = 0, \quad \&c. \ \&c.;$$

therefore, by eliminating $(\mu, \lambda, -m, -l)$, whatever l, m, λ, μ may be, we shall have for *one* arrangement that given in the Question, provided only that we have the relation $\dfrac{I'^3}{J'^2} = \dfrac{I^3}{J^2}$, obtained by eliminating M.

1864. (Proposed by the Rev. J. BLISSARD.)—Prove that

$$(1) \ldots 1 - n + \frac{n(n-1)}{1.2} \ldots \pm \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r} = \pm \frac{(n-1)(n-2)\ldots(n-r)}{1.2\ldots r},$$

$$(2) \ldots \frac{1}{m+1} \ldots + \frac{1}{m+n} = \frac{n(n+1)\ldots(n+m)}{1.2\ldots m} \left\{ \frac{1}{(m+1)^2} - \frac{n-1}{1} \cdot \frac{1}{(m+2)^2} + \&c. \right\}$$

Solution by SAMUEL ROBERTS, M.A.

Since $(1-x)^{n-1} = (1-x)^n (1 + x + \ldots + x^r) + x^{r+1}(1-x)^{n-1}$, we get the first result by equating coefficients of x^r. Also

$$1 = \text{Co. of } x^{n-1} \text{ in } (1+x)^{n-1} = \text{Co. of } x^{n-1} \text{ in } (1+x)^{m+n}(1+x)^{-(m+1)}$$

$$= \frac{\Gamma(m+n+1)}{\Gamma(n)\,\Gamma(m+2)} - (m+1)\frac{\Gamma(m+n+1)}{\Gamma(n-1)\,\Gamma(m+3)} + \ldots$$

$$= \frac{\Gamma(m+n+1)}{\Gamma(n)\,\Gamma(m+1)} \left\{ \frac{1}{m+1} - \frac{n-1}{1} \cdot \frac{1}{m+2} + \ldots \right\},$$

therefore $$\frac{\Gamma(m+1)}{\Gamma(m+n+1)} = \frac{1}{\Gamma(n)} \left\{ \frac{1}{m+1} - \frac{n-1}{1} \cdot \frac{1}{m+2} + \ldots \right\};$$

and differentiating with regard to m, and writing $\dfrac{(m+n)(m+n-1)\ldots n}{1.2\ldots m}$

for $\frac{(m+n)\,..\,(m+1)}{1\,.\,2\,..\,(n-1)}$, we have the second result, which however is more directly obtained by use of $\int_0^1 z^m (1-z)^{n-1}\, dz = \frac{\Gamma(n)\,\Gamma(m+1)}{\Gamma(m+n+1)}$.

1810. (Proposed by C. TAYLOR, M.A.)—Prove that the value of the expression $\{\cos(\alpha-\beta)-\cos(\gamma-\delta)\}^2+\{\cos(\alpha+\beta)-\cos(\gamma+\delta)\}^2+\{\cos(\alpha+\gamma)\cos(\beta-\delta)-\cos(\beta+\delta)\cos(\alpha-\gamma)\}^2$ is unaltered by the interchange of β, γ.

Solution by J. WALMSLEY.

Writing a^2, b^2, c^2 for the terms of the given expression, and C for the sum of certain quantities not affected in value by the interchange of β, γ, we have

$$a^2+b^2 = 4\sin^2\tfrac{1}{2}(\alpha-\beta+\gamma-\delta)\sin^2\tfrac{1}{2}(\alpha-\beta-\gamma+\delta)$$
$$+4\sin^2\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)\sin^2\tfrac{1}{2}(\alpha+\beta-\gamma-\delta)$$
$$= 2\sin^2\tfrac{1}{2}(\alpha-\beta-\gamma+\delta)\left\{1-\cos(\alpha-\delta)\cos(\beta-\gamma)-\sin(\alpha-\delta)\sin(\beta-\gamma)\right\}$$
$$+2\sin^2\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)\left\{1-\cos(\alpha-\delta)\cos(\beta-\gamma)+\sin(\alpha-\delta)\sin(\beta-\gamma)\right\}$$
$$= C+2\sin(\alpha-\delta)\sin(\beta-\gamma)\sin(\alpha+\delta)\sin(\beta+\gamma);$$
$$c^2 = \tfrac{1}{4}\left\{\cos(\alpha+\beta+\gamma-\delta)+\cos(\alpha-\beta+\gamma+\delta)-\cos(\alpha+\beta-\gamma+\delta)\right.$$
$$\left.-\cos(\alpha-\beta-\gamma-\delta)^2\right\} = \left\{\sin(\alpha+\delta)\sin(\beta-\gamma)-\sin(\alpha-\delta)\sin(\beta+\gamma)\right\}^2;$$

therefore the given expression becomes

$$C+\sin^2(\alpha+\delta)\sin^2(\beta-\gamma)+\sin^2(\alpha-\delta)\sin^2(\beta+\gamma),$$

and is hence not altered in value by the proposed interchange.

1822. (Proposed by W. S. BURNSIDE, B.A.)—Prove that the area of a triangle circumscribing a conic is $ab^{-2}p_1p_2p_3$, where p_1 is one of the four perpendiculars from the vertex (1) on the focal vectors to the points of contact of tangents from the same vertex.

Solution by W. H. LAVERTY; R. WARREN, B.A.; J. DALE; *and others*.

Let a, b be the semi-axes of the conic, and α, β, γ the eccentric angles of the points of contact of the sides 23, 31, 12, respectively; then it may be readily shown that the lengths of the perpendiculars on the focal vectors to these points, drawn from the opposite vertices 1, 2, 3, are, respectively,

$$p_1 = b\tan\tfrac{1}{2}(\beta-\gamma),\quad p_2 = b\tan\tfrac{1}{2}(\gamma-\alpha),\quad p_3 = b\tan\tfrac{1}{2}(\alpha-\beta).$$

But (Salmon's *Conics*, Art. 231, ex. 9) the area of the triangle 123 is

$$ab\tan\tfrac{1}{2}(\alpha-\beta)\tan\tfrac{1}{2}(\beta-\gamma)\tan\tfrac{1}{2}(\gamma-\alpha),\text{ or } ab^{-2}p_1p_2p_3.$$

Note on the Problems in regard to a Conic defined by five Conditions of Intersection. By Professor Cayley.

I use the word "intersection" rather than "contact," because it extends to the case of a 1-pointic intersection, which cannot be termed a contact. The conditions referred to are that the conic shall have with a given curve, at a point given or not given, a 1-pointic intersection, a 2-pointic intersection (= ordinary contact), a 3-pointic intersection, &c., as the case may be. It may be noticed that when the point on the curve is a given point, the condition of a k-pointic intersection is really only the condition that the conic shall pass through k given points; though from the circumstance that these are consecutive points on a conic, the formulæ for a conic passing through k discrete points require material alteration; for instance, in the two questions to find the equation of a conic passing through five given points, and to find, the equation of a conic having at a given point of a given curve 5-pointic intersection with the curve, the forms of the solutions are very different from each other.

The foregoing remark shows, however, that it is proper to detach the conditions which relate to intersections at given points; and consequently attending only to the conditions which relate to intersection at an unascertained point (of course the intersections referred to must be at least 2-pointic, for otherwise there is no condition at all) we may consider the conics which pass through four points and satisfy one condition; or which pass through three points and satisfy two conditions; or which pass through two points and satisfy three conditions; or which pass through one point and satisfy four conditions; or which satisfy five conditions. Considering in particular the last case, let 1 denote that the conic has 2-pointic intersection, 2 that it has 3-pointic intersection, ... 5 that it has 6-pointic intersection with a given curve at an unascertained point.

Then the problems are in the first instance

5 ; 4, 1 ; 3, 2; 3, 1, 1 ; 2, 2, 1; 2, 1, 1, 1; 1, 1, 1, 1, 1.

But the intersections may be intersections with the same given curve or with different given curves; and we have thus in all 27 problems viz., these are as given in the following table, where the colons (:) separate those conditions which refer to different curves:—

No. of Prob.	Conditions.	No. of Prob.	Conditions.	No. of Prob.	Conditions.
1	5	10	3, 1 : 1	19	3 : 1 : 1
2	4, 1	11	3 : 1, 1	20	2 : 2 : 1
3	3, 2	12	2, 2 : 1	21	2, 1 : 1 : 1
4	3, 1, 1	13	2, 1 : 2	22	2 : 1, 1 : 1
5	2, 2, 1	14	2, 1, 1 : 1	23	1, 1, 1 : 1 : 1
6	2, 1, 1, 1	15	2, 1 : 1, 1	24	1, 1 : 1, 1 : 1
7	1, 1, 1, 1, 1	16	2 : 1, 1, 1	25	2 : 1 : 1 : 1
8	4 : 1	17	1, 1, 1, 1 : 1	26	1, 1 : 1 : 1 : 1
9	3 : 2	18	1, 1, 1 ; 1, 1	27	1 : 1 : 1 : 1 : 1

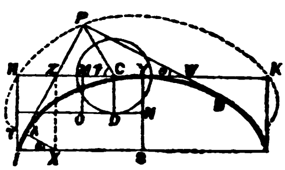
Thus Problem 1 is to find a conic having 6-pointic intersection with a given curve; Problem 2 a conic having 5-pointic intersection and also 2-pointic intersection with a given curve..... Problem 7 is to find a conic having five 2-pointic intersections with (touching at five distinct points) a given curve. .... Problem 27 is to find a conic having 2-pointic intersection with

(touching) each of five given curves. Or we may in each case take the problem to be merely to find the number of the conics which satisfy the required conditions. This number is known in Prob. 1, for the case of a curve of the order m without singularities, viz., the number is $= m(12m-27)$. It is also known in Problems 25 and 26 in the case where the first curve (that to which the symbol 2, or 1, 1 relates) is a curve without singularities; and it is known in Prob. 27, viz., if m, n, p, q, r be the orders and M, N, P, Q, R the classes of the five curves respectively, then the number is $= (M, m)(N, n)(P, p)(Q, q)(R, r)\{1, 2, 4, 4, 2, 1\}$, that is, $1\,MNPQR + 2\,\Sigma MNPQr + \&c.$ The number is not, I believe, known in any other of the problems. In particular, (Prob. 7) we do not as yet know the number of the conics which touch a given curve at five points. It would be interesting to obtain this number; but (judging from the analogous question of finding the double tangents of a curve) the problem is probably a very difficult one.

1694. (Proposed by M. W. CROFTON, B.A.)—Show that if tangents at right angles are drawn to a cycloid, the locus of their intersection is a trochoid, whose generating circle is equal to that of the cycloid (radius r) and rolls along the line which bisects the height of the cycloid at right angles; the distance of the describing point from the centre being $\frac{1}{2}\pi r$. Show how the *loops* of this trochoid form part of the locus.

Solution by the PROPOSER; J. DALE; *and others.*

Lemma.—It is easily proved that, if PA, PB be two perpendicular tangents to a cycloid, the portion ZW which they intercept on HK is always equal to πr.



Bisect now ZW in C, then $CP = \frac{1}{2}\pi r$; draw the normal AX, then XZ is parallel to YS, the height. Bisect HY in M, then $\frac{1}{2}\pi r = HM = CZ$; hence $MC = HZ = IX = 2r\theta$: but PCZ (or ψ) $= 2\theta$, therefore $MC = r\psi$.

Describe a circle from C as centre with radius r; it touches ON in D (N being the middle point of YS); now, supposing this circle to have rolled along the line OD, from O, the angle through which it has turned is $\frac{OD}{r} = \frac{MC}{r} = \psi$; hence, if we suppose the line MH to have been rigidly connected with it in its first position, when its centre was at M, it will now have carried the point H from its original position to P; for $CP = MH$. Hence the locus sought is a trochoid THPK. The *loops* of this trochoid, below H and K, will be found to be the loci of the intersections of tangents to the cycloid with perpendicular tangents to the preceding and succeeding cycloids.

1778. (Proposed by W. S. Burnside, B.A.)—If the line given by the equation $ax + \beta y + \gamma z = 0$ intersect the conic $(a, b, c, f, g, h)(x, y, z)^2 = 0$ in the points P, Q; the tangents to the conic at these points meeting in the point R, and a focus being at the point F: prove that $\dfrac{FP \, . \, FQ}{FR^2} = \dfrac{\Pi^2}{\Pi^2 - \Theta\Sigma}$,

where $\quad \Theta = (A, B, C, F, G, H)(\sin A, \sin B, \sin C)^2$,

$$\Sigma = (A, B, C, F, G, H)(\alpha, \beta, \gamma)^2, \quad 2\Pi = \sin A \frac{d\Sigma}{d\alpha} + \sin B \frac{d\Sigma}{d\beta} + \sin C \frac{d\Sigma}{d\gamma}.$$

Solution by the Proposer.

1. If ϕ and ϕ_1 are the eccentric angles of the points P and Q; then $\tan^2 \frac{1}{2}(\phi - \phi_1) = -\Theta\Sigma\Pi^{-2}$. For if $ax + \beta y + \gamma z$ and $\lambda X + \mu Y + \nu$ be identical, where $a\lambda = \cos\frac{1}{2}(\phi + \phi_1)$, $b\mu = \sin\frac{1}{2}(\phi + \phi_1)$, $\nu = -\cos\frac{1}{2}(\phi - \phi_1)$; we have $\Sigma \equiv q\,(a^2\lambda^2 + b^2\mu^2 - \nu^2) \equiv q \sin^2 \frac{1}{2}(\phi - \phi_1)$, whence $\Theta \equiv -qM$ (where $M = x \sin A + y \sin B + z \sin C$), and $\Pi \equiv -qM\nu \equiv qM \cos\frac{1}{2}(\phi - \phi_1)$; hence, finally, $-\Theta\Sigma\Pi^{-2} = \tan^2 \frac{1}{2}(\phi - \phi_1)$.

2. To prove that $FP \, . \, FQ = FR^2 \cos^2 \frac{1}{2}(\phi - \phi_1)$.

We have $\quad FP = a(1 - e\cos\phi)$, and $FQ = a(1 - e\cos\phi_1)$;

whence $FP \, . \, FQ = a^2 \left\{\cos\frac{1}{2}(\phi + \phi_1) - e \cos\frac{1}{2}(\phi - \phi_1)\right\}^2 + b^2 \sin^2\frac{1}{2}(\phi + \phi_1)$.

But the coordinates of R are $\dfrac{x}{a} = \dfrac{\cos\frac{1}{2}(\phi + \phi_1)}{\cos\frac{1}{2}(\phi - \phi_1)}$, $\dfrac{y}{b} = \dfrac{\sin\frac{1}{2}(\phi + \phi_1)}{\cos\frac{1}{2}(\phi - \phi_1)}$;

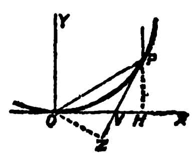
$$\therefore \; FP \, . \, FQ = \left\{(x - c)^2 + y^2\right\} \cos^2 \tfrac{1}{2}(\phi - \phi_1) = FR^2 \, . \cos^2 \tfrac{1}{2}(\phi - \phi_1).$$

Hence $\quad \dfrac{FP \, . \, FQ}{FR^2} = \cos^2 \frac{1}{2}(\phi - \phi_1) = \dfrac{1}{1 + \tan^2 \frac{1}{2}(\phi - \phi_1)} = \dfrac{\Pi^2}{\Pi^2 - \Theta\Sigma}$.

3. Hence if $FP \, . \, FQ : FR^2 = F_1P_1 \, . \, F_1Q_1 : F_1R_1^2$ where two concentric conics are cut by the line $ax + \beta y + \gamma z = 0$, the envelope of this line is the *parabola* $\Theta\Sigma' - \Theta'\Sigma = 0$. For the above relation gives $\Pi^2 : \Theta\Sigma = \Pi'^2 : \Theta'\Sigma'$, and since the conics are concentric, $\Pi : \Theta = \Pi : \Theta'$.

1682. (Proposed by M. W. Crofton, B.A.)—Find a curve such that the radii of curvature at any two points are to each other as the cubes of the tangents drawn from those points to meet each other.

Solution by the Proposer.



Refer the curve to a fixed tangent and normal, OX, OY; let a be the radius of curvature at O, ρ that at the point P;

then $\quad \dfrac{\rho}{a} = \dfrac{PV^3}{OV^3} = \dfrac{PH^3}{OZ^3} = \dfrac{y^3}{p^3}$;

whence we obtain $\dfrac{dr}{r^2} = a \dfrac{dp}{p^3} \sin^3\theta$;

or putting $u = \frac{1}{r}$, we have, as $\frac{1}{p^2} = u^2 + \frac{du^2}{d\theta^2}$,

$$\frac{du}{d\theta} = a\frac{du}{d\theta}\sin^3\theta\left(u + \frac{d^2u}{d\theta^2}\right), \qquad \text{hence } \frac{d^2u}{d\theta^2} + u = \frac{1}{a\sin^3\theta},$$

the integral of which is $au = \sin\theta\int\frac{\cos\theta}{\sin^3\theta}\,d\theta - \cos\theta\int\frac{d\theta}{\sin^2\theta}$,

or
$$au = -\frac{1}{2\sin\theta} + \frac{\cos^2\theta}{\sin\theta} + C_1\sin\theta + C_2\cos\theta,$$

the equation of the curve sought: or, returning to rectangular coordinates,

$$ay = \tfrac{1}{2}x^2 + Ay^2 + Bxy,$$

A and B being arbitrary constants: hence, the only curves which possess the property are the conic sections.

[It may be shown as follows that in the conic $ax^2 + bxy + cy^2 = dy$ the radius of curvature at the origin is $\frac{d}{2a}$; for this equation may be written $a(x^2+y^2) - dy = (a-c)y^2 - bxy$; hence the circle $a(x^2+y^2) = dy$ cuts the conic on the lines $y = 0$, $(a-c)y = bx$, so that three of the intersections coincide at the origin; hence this is the osculating circle there.]

1826. (Proposed by G. O. Hanlon.)—The vertex of a triangle is fixed, while its base, of constant length, moves along a given line; show that the locus of the centre of the circumscribed circle is a parabola.

Solution by R. Warren, B.A.; *the* Rev. J. L. Kitchin, M.A.; J. Dale; W. H. Laverty; *the* Proposer; *and others.*

Take the fixed line on which the base moves as axis of y, and the perpendicular to it through the vertex as axis of x. Let $2c$ be the length of the base; and $(a, 0)$, (x, y) the respective coordinates of the vertex and the centre of the circumscribed circle.

Then obviously the equation of the required locus is

$$(a-x)^2 + y^2 = x^2 + c^2, \text{ or } y^2 = 2ax - (a^2 - c^2),$$

which is that of a parabola, whose parameter is $2a$ and whose vertex is situated on the axis of x at a distance of $\frac{a^2-c^2}{2a}$ from the origin.

[Another investigation of the locus is given in the *Note* to Quest. 1506, *Reprint*, Vol. II., p. 46; and the *envelope* of the series of circles is determined in the Solution to that Question.]

1827. (Proposed by W. H. LAVERTY.)—Find the values of x, y, z which make the function $u = f(x) \,.\, \phi(y) \,.\, \psi(z)$ a maximum; x, y, z being connected by the equation $a^{f(x)-\alpha} \,.\, b^{\phi(y)-\beta} \,.\, c^{\psi(z)-\gamma} = \text{A}$.

Solution by the REV. J. L. KITCHIN, M.A.; R. WARREN, B.A.; *the* PROPOSER; *and others.*

Differentiating function and condition, we have

$$\frac{f'(x)}{f(x)} \,.\, dx + \frac{\phi'(y)}{\phi(y)} \,.\, dy + \frac{\psi'(z)}{\psi(z)} \,.\, dz = 0,$$

$$f'(x) \,.\, dx \,.\, \log a + \phi'(y) \,.\, dy \,.\, \log b + \psi'(z) \,.\, dz \,.\, \log c = 0.$$

$$\text{Let } \frac{1}{f(x)} = k \,.\, \log a; \quad \therefore \frac{1}{\phi(y)} = k \,.\, \log b; \quad \text{and } \frac{1}{\psi(z)} = k \,.\, \log c,$$

$$\therefore\ 3 = k \left\{ \log a^{f(x)} + \log b^{\phi(y)} + \log c^{\psi(z)} \right\} = k \log (\text{A} \,.\, a^{\alpha} \,.\, b^{\beta} \,.\, c^{\gamma}),$$

$$\therefore f(x) = \frac{\log (\text{A} \,.\, a^{\alpha} \,.\, b^{\beta} \,.\, c^{\gamma})}{3 \log a}; \quad \phi(y) = \frac{\log (\text{A} \,.\, a^{\alpha} \,.\, b^{\beta} \,.\, c^{\gamma})}{3 \log b}; \quad \psi(z) = \&\text{c.},$$

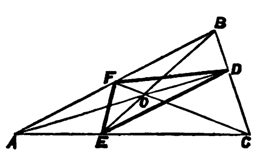
which give equations for determining x, y, z; also the maximum value is

$$u = f(x) \,.\, \phi(y) \,.\, \psi(z) = \frac{\left\{ \log (\text{A} \,.\, a^{\alpha} \,.\, b^{\beta} \,.\, c^{\gamma}) \right\}^3}{\log a^3 \,.\, \log b^3 \,.\, \log c^3}.$$

1683. (Proposed by R. TUCKER, M.A.)—Of all the concurrently-connectant triangles that can be inscribed in a given triangle, prove that the maximum is that for which the concurrent point is the centroid of the given triangle. Find also the maximum triangle when the concurrent point lies on the perpendicular from one angle on the opposite side, or on the bisector of the angle; and find, furthermore, the locus of the concurrent point when the area of the connectant triangle is *constant.*

Solution by the PROPOSER; E. McCORMICK; E. FITZGERALD; J. DALE; *and others.*

1. Take AC, AB, for axes, then the area of the triangle DEF, corresponding to the concurrent point O, or (x, y), will be



$$\text{since } d \text{ is } \left(\frac{bcx}{by + cx}, \frac{bcy}{by + cx} \right)$$

$$e \text{ is } \left(\frac{cx}{c - y}, 0 \right)$$

$$f \text{ is } \left(0, \frac{by}{b - x} \right)$$

$$bc \sin \text{A} \,(-u), \text{ where } u = \frac{xy\,(by + cx - bc)}{(b - x)\,(c - y)\,(by + cx)} \cdots\cdots\cdots\cdots (a).$$

Now $\dfrac{du}{dx} = \dfrac{b^2y^2(by+2cx-bc)}{(b-x)^2(c-y)(by+cx)^2}$, $\dfrac{du}{dy} = \dfrac{c^2x^2(cx+2by-bc)}{(b-x)(c-y)^2(by+cx)^2}$;

hence for a maximum or minimum

$$by+2cx = bc = cx+2by, \text{ or } x = \tfrac{1}{3}b,\ y = \tfrac{1}{3}c,$$

therefore O is the *centroid* of the triangle.

For the above point we find

$$\frac{d^2u}{dx^2} = \frac{27}{8b^2},\ \frac{d^2u}{dy^2} = \frac{27}{8c^2},\ \frac{d^2u}{dx\,dy} = \frac{27}{32bc};$$

whence we see that the triangle is a *maximum*.

2. Again, we have $\dfrac{dy}{dx} = -\dfrac{b^2y^2(c-y)(by+2cx-bc)}{c^2x^2(b-x)(cx+2by-bc)}$; hence when O is restricted to the perpendicular from A on BC, whose equation is $x\cos C = y\cos B$, we have

$$c^2x^2(b-x)(cx+by-bc)\cos C + b^2y^2(c-y)(by+2cx-bc)\cos B = 0,$$

or $$(a^2-bc\cos B\cos C)\,x^2-2(abc\cos B)\,x+b^2c^2\cos^2 B = 0;$$

$$\therefore\ x = \frac{bc\cos B}{a\pm(bc\cos B\cos C)^{\frac{1}{2}}};\quad \text{maximum area} = \frac{bc^2\sin B\cos B\cos C}{a\pm2(bc\cos B\cos C)^{\frac{1}{2}}}$$

3. In the case when O is restricted to the bisector of the angle A, we have, since the equation to this line is $y = x$,

$$b^2x^2(c-x)(bx+2cx-bc)+c^2x^2(b-x)(cx+2bx-bc) = 0,$$

or $$(b+c)(b^2+bc+c^2)\,x^2-2bc\,(b+c)^2\,x+b^2c^2(b+c) = 0,$$

whence $x = \dfrac{bc}{b\mp\sqrt{bc}+c}$; maximum area $= \dfrac{b^2c^2\sin A}{(b+c)(\sqrt{b}\mp\sqrt{c})^2}$.

4. From equation (α) above we see that if the triangle DEF is of constant area (viz. $k\triangle ABC = \frac{1}{2}kbc\sin A$) and the point O unrestricted in position, the locus of O will be a cubic given by the equation

$$2xy\,(by+cx-bc)+k\,(b-x)(c-y)(by+cx) = 0\ldots\ldots\ldots.(\beta).$$

If now we write $c = mb$, and arrange, we get

$$y^2-m\,(b-x)\,y = -\frac{m^2bkx\,(b-x)}{2x-k\,(b-x)},$$

whence $$y = \frac{m\,(b-x)}{2}\pm\frac{m\,(b-x)^{\frac{1}{2}}\left\{2x(b-x)-k\,(b+x)^2\right\}^{\frac{1}{2}}}{2\left\{2x-k\,(b-x)\right\}^{\frac{1}{2}}}.$$

The asymptotes are given by the equations

$$y = \frac{kc}{2+k},\quad x = \frac{kb}{2+k},\quad y = -mx+\frac{2c}{2+k}.$$

The abscissas corresponding to the tangential ordinates of the *oval* will be given by the consideration that the two values of y must coincide, in which case we readily see that

$$2x\,(b-x)-k\,(b+x)^2 = 0,\ \text{or}\ x = b\left\{\frac{1-k\pm(1-4k)^{\frac{1}{2}}}{2+k}\right\}.$$

From these values of x we see that when $k = \frac{1}{4}$ the oval becomes a point, viz. the centroid.

Again, if y_1, y_2 be ordinates corresponding to the same abscissa x, we have $y_1+y_2 = m(b-x)$, which shows that the oval cuts an ordinate produced to meet BC in such a way that the intercepts outside it are equal.

Since x may have any negative value or any positive value greater than b, we see that the curve is of the following nature:—

$k < \frac{1}{4}$, three infinite asymptotic branches passing through A, B, C, and an oval interior to the triangle ABC;

$k = \frac{1}{4}$, the oval reduces to the centroid;

$k > \frac{1}{4}$, three infinite asymptotic branches only.

The maximum and minimum ordinates correspond to the abscissas found from the equation $k(b-x)^2 + x(2x-b) = 0$, obtained by putting $\frac{dy}{dx} = 0$,

whence $$x = \frac{b}{2}\left\{\frac{1+2k \pm (1-4k)^{\frac{1}{2}}}{2+k}\right\}.$$

[*Otherwise:* $\frac{\Delta\,\text{OEF}}{\Delta\,\text{OBC}} = \frac{\text{OE}\,.\,\text{OF}}{\text{OB}\,.\,\text{OC}}$, $\frac{\text{OBC}}{\text{ABC}} = \frac{\text{OD}}{\text{AD}}$, $\therefore \frac{\text{OEF}}{\text{ABC}} = \frac{\text{OD}\,.\,\text{OE}\,.\,\text{OF}}{\text{OB}\,.\,\text{OC}\,.\,\text{AD}}$;

$$\therefore \frac{\text{DEF}}{\text{ABC}} = \frac{\text{OD}\,.\,\text{OE}\,.\,\text{OF}}{\text{OA}\,.\,\text{OB}\,.\,\text{OC}}\left(\frac{\text{OA}}{\text{AD}} + \frac{\text{OB}}{\text{BE}} + \frac{\text{OC}}{\text{CF}}\right) = 2\,\frac{\text{OD}\,.\,\text{OE}\,.\,\text{OF}}{\text{OA}\,.\,\text{OB}\,.\,\text{OC}}.$$

Let DEF $= k$, ABC $= 1$, BOC $= x$, COA $= y$, AOB $= z$; then we have
$$k = \frac{2xyz}{(y+z)(z+x)(x+y)} = \frac{2xyz}{yz+zx+xy-xyz};$$

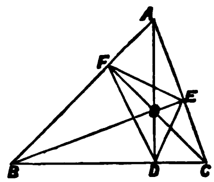
therefore $1 + \frac{2}{k} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ ($= u$ suppose); also $x+y+z = 1$.

Now u is readily found to be a *minimum* ($u = 9$) when $x = y = z = \frac{1}{3}$, that is to say when O is the centroid of the triangle ABC; hence k, that is the triangle DEF, is then a *maximum* ($k = \frac{1}{4}$). When the area (k) of the triangle DEF is *constant*, the equation, in triangular coordinates (x, y, z) of the locus of O is the cubic
$$2xyz = k(y+z)(z+x)(x+y).$$

This cubic has been discussed at some length, with several illustrative figures for different values of the parameter, by Mr. WHITWORTH, in a very interesting paper in the *Messenger of Mathematics*, (Vol. II., pp. 123—127). The properties there investigated by *trilinear* coordinates agree with those which Mr. TUCKER has developed above by *Cartesian* coordinates: thus, for instance, the asymptotes are given by equations which we may interpret by saying that these lines are parallel to the sides of the given triangle ABC, and divide each of the sides which they cut so that the segment towards the parallel side is $\frac{1}{2}k$ times that towards the opposite angle.]

1782. (Proposed by J. WILSON.)—According as one of a triangle is a geometric, harmonic, or arithmetic mean between the other two, so is the cosine of one of the semi-angles of the *pedal triangle* a geometric, harmonic, or arithmetic mean between the cosines of the semi-angles of the other two.

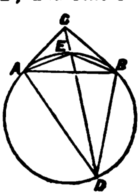
Solution by E. CONOLLY; E. MCCORMICK; E. FITZGERALD; J. DALE; H. MURPHY; *and many others.*



Let ABC be the triangle, DEF the *pedal triangle.* Then, since ABDE, ACDF are inscriptible in circles, the angle EDC = BAC = FDB, &c.; whence it follows that the semi-angles of the *pedal triangle* are the complements of the angles of the original triangle; the cosines of these semi-angles are, therefore, proportional to the sides on which their vertices lie. Whatever homogeneous relations, therefore, subsist between the sides of the original triangle, the same must subsist between the cosines of the semi-angles of the *pedal triangle.*

1762. (Proposed by J. O'CALLAGHAN.)—From the intersection of two tangents to a circle to draw a line, cutting the circumference in two points such that, if they are joined with the points of contact of the tangents, the rectangle contained by either pair of the opposite sides of the quadrilateral thus formed may be given or a maximum.

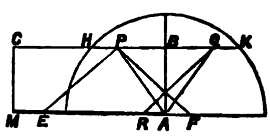
Solution by E. MCCORMICK; H. MURPHY; E. CONNOLLY; *and others.*



Let CA, CB be the two tangents, and CED the line required. Then, from the similarity of the triangles AEC and DAC, BEC and DBC,
we have DA : AE = DC : CA (or CB) = DB : BE,
therefore DA . BE = DB . AE = $\frac{1}{2}$AB . DE, by Ptolemy's theorem; and hence the solution of the problem is manifest.

1177. (Proposed by S. WATSON.)—Two rods, connected at their middle points by a string whose length exceeds half the sum of the lengths of the rods, are thrown at random on the ground; what is the chance that they will rest across each other?

Solution by the PROPOSER.



Let $2a$ be the length of the longer rod, $2b$ that of the shorter rod, and $2s$ ($>a+b$) the length of the string. Let AM represent half the longer rod, A being one end and M the middle of the rod. Draw BC parallel to AM, and AB, MC perpendicular

to AM. Take in CB any two points P, Q equidistant from B; draw PE = PF = QR = b, and join AP, AQ. Also with A as centre and radius b draw a circle cutting BC in H, K. Put $\angle$EPF = 2θ, then AB = MC = $b\cos\theta$. Now $\angle$ AQR = APF, therefore EPA + AQR = EPF = 2θ. Hence we may always consider the middle of the shorter rod to lie between B and C, and to cross the longer rod while either end revolves through an angle 2θ. Hence doubting twice, first on account of the other half of the longer rod, and next because BC may lie on either side of AM, we obtain for the measure of the positions favourable for crossing

$$F = 16ab\int \theta d\,(b\cos\theta) = 16ab^2\int_0^{\frac{1}{2}\pi}\theta\sin\theta\, d\theta = 16ab^2.$$

The measure of the whole number of positions is $W = (2\pi b)(4\pi s^2) = 8\pi^2 bs^2$: hence the required chance is

$$p = \frac{F}{W} = \frac{2ab}{\pi^2 s^2};\ \text{which, if } a=b=s, \text{ becomes } p = \frac{2}{\pi^2}.$$

[When one of the rods is half the length of the other, and the string half the length of the shorter rod, it is shown, in the *Lady's and Gentleman's Diary* for 1860, Quest. 1958, that the chance of crossing is $\frac{1}{2}+\frac{2}{\pi^2}$].

Solution of the Problem in Vol. II., p. 74, of the Reprint.

By Professor De Morgan.

A straight line being divided at random in two places; find the chance that the triangle formed by the three parts has all its angles acute.

The very peculiar method adopted by the Editor induced me to try whether a more direct method would offer any difficulty.

Let the whole line ($2a$) be filled with points: the number of points in a line being proportional to its length. When I say the number of points in a line is (as) x, I mean that it is $\infty\,.\,x$, the symbol ∞ being of one *value* in all lines. Thus dx contains $\infty\,.\,dx$ points, at *subequal* distances from the commencement of x. I call two magnitudes *subequal* which differ by an infinitely small part of either.

First, how many ways are there of dividing $2a$ into three parts? Answer, $\frac{1}{2}\infty\,.\,2a\,(\infty\,.\,2a-1)$ or $\infty^2\,.\,2a^2$; say as $2a^2$. This may be verified by

$$\int_0^{2a}\int_x^{2a} dx\,dy.$$

Next, how many of these ways give triangles? Let x, y, z be the consecutive divisions of $2a$; let x be the greatest, and z the least. Six times the number of such divisions is the answer required. Now x must lie between $\frac{2}{3}a$ and a, since $y+z>x$; and y must lie between $\frac{1}{2}(2a-x)$ and x; and

$$\int_{\frac{2}{3}a}^{a}\int_{\frac{1}{2}(2a-x)}^{x} dx\,dy = \tfrac{1}{12}a^2.$$

Hence $\frac{1}{2}a^2$ is (as) the number of triangles; or the chance of getting a triangle is $\frac{1}{4}$.

The number of acute-angled triangles is six times the number of cases in which $x > y > z$, and $y^2+z^2 > x^2$, or

$$y^2+(2a-x-y)^2 > x^2, \text{ or } y^2-(2a-x)\,y+2a\,(a-x) > 0.$$

This expression ($x < a$) has two positive roots, when x is great enough; their sum is $2a-x$, so we must take the greater for the lowest value of y.

It is $$\tfrac{1}{2}\left\{2a-x+\sqrt{(2a+x)^2-8a^2}\right\} = t.$$

So long as the roots are imaginary, we must integrate dy from $\frac{1}{2}(2a-x)$ to x; but when the roots become real, we must integrate from t to x. This amounts to $\int_{\frac{2}{3}a}^{a}\int_{t}^{x} dx\,dy$, with the imaginary part of the result omitted.

Now $$\int_{t}^{x} dy = \tfrac{1}{2}\left\{3x-2a-\sqrt{(2a+x)^2-8a^2}\right\},$$

and $$\int_{\frac{2}{3}a}^{a}\int_{t}^{x} dx\,dy = \left\{-\tfrac{2}{3}+\tfrac{2}{3}\sqrt{\left(-\tfrac{8}{9}\right)}-2\log\left(\tfrac{2}{3}+\sqrt{-\tfrac{1}{18}}\right)\right\}a^2.$$

Of this the real part is $(-\frac{2}{3}-\log\frac{1}{2})\,a^2$; and six times this is $(6\log 2-4)a^2$; which is (as) the number of acute-angled triangles. And $2a^2$ being (as) the whole number of triangles, the chance of an acute-angled triangle is

$$3\log 2-2.$$

This agrees with the result of the Editor, whose ingenuity I greatly admire: but I doubt if any soul alive would fully believe either of us, if it were not for the other.

The Problem—A line being divided at hazard into n parts, required the chance of a polygon—offers no difficulty of translation into a multiple integral; I should like to see the result of the integration.

[The Editor obtained the result at first by the aid of the Integral Calculus, and his Solution by this method was published in the *Educational Times* for November, 1859. In the next Number however (the Editorship being then in other hands) the correctness of this method and result was called in question by a correspondent, who substituted instead an erroneous solution, leading to a different result. The present Editor thereupon, in reply, drew up an article, in which, after pointing out the errors in the second Solution, he gave a geometrical solution in corroboration of the result obtained by his former method. This last solution was published in the *Educational Times* for January, 1860, and afterwards reproduced with some alterations in the volume of the *Reprint* referred to by Professor De Morgan. The foregoing remarks are offered here in explanation of the peculiarity of the method.

The Problem enunciated at the end of Professor De Morgan's Solution will be found proposed, with the *result* annexed, as Question 1878.]

1887. (Proposed by Professor SYLVESTER.)—Find the mean value of the volume of a tetrahedron, three of whose vertices lie respectively in three non-intersecting edges, and the fourth at the centre of a given parallelepiped.

Solution by the PROPOSER.

It may easily be proved that the proportion of the mean tetrahedron in question to the volume of the parallelepiped is independent of the form and dimensions of the latter. For greater simplicity, then, we may suppose it to be a cube whose sides are each 2. Take the principal axes of the cube through the centre as the coordinate axes: then, calling x, y, z the distances of the variable points from the centres of their respective ranges, the volume of the tetrahedron is $\dfrac{\mathrm{V}}{6}$, where $\mathrm{V} = \pm \begin{vmatrix} x, & 1, & -1 \\ -1, & y, & 1 \\ 1, & -1, & z \end{vmatrix} = \pm\,(xyz + x + y + z)$, with the understanding that V is to be always *positive*.

There will be 8 cases reduceable to a duplication of the four following; viz., x, y, z, all positive, or all but one positive.

When x, y, z are all positive, V, operated upon by

$$\int_0^1 dz \int_0^1 dy \int_0^1 dx \text{ gives } \tfrac{13}{8}, \text{ as is easily found.}$$

When x is negative, say $-\xi$, the positive value of V is $-\xi yz - \xi + y + z$ from $\xi = 0$ to $\xi = \dfrac{y+z}{yz+1}$ say (ξ), provided the latter is not greater than unity. This condition is necessarily fulfilled, for $1 - (\xi) = \dfrac{(1-y)(1-z)}{yz+1}$, which, since y, z are each included between 0, 1, is necessarily positive. From $\xi = (\xi)$ to $\xi = 1$ the positive value of V is $\xi yz + \xi - y - z$. Hence the derived integral may be found by operating upon this latter quantity with $\int_0^1 dz \int_0^1 dy \int_0^1 d\xi$, and adding the result of operating upon $-\xi yz - \xi + y + z$ with $2\int_0^1 dz \int_0^1 dy \int_0^{(\xi)} d\xi$. The first operate is $-\tfrac{3}{8}$.

Again, the result of the operation $2\int_0^{(\xi)} d\xi$ upon its operand is

$$\frac{(y+z)^2}{yz+1} = \frac{y^2}{yz+1} + \frac{z^2}{yz+1} - \frac{2}{yz+1} + 2.$$

Hence the result of the complete operation will be $2 - 2\mathrm{P} + 2\mathrm{Q}$, where

$$\mathrm{P} = \int_0^1 \int_0^1 dy\,dz\, \frac{1}{1+yz} = \int_0^1 dy\, \frac{\log(1+y)}{y} = 1 - \frac{1}{2^2} + \frac{1}{2^2} - \frac{1}{4^2} \ldots = \frac{\pi^2}{12},$$

$$\mathrm{Q} = \int_0^1 \int_0^1 dy\,dz\, \frac{y^2}{1+yz} = \int_1^2 d\omega\,(\omega - 1)\log\omega = \frac{1}{4}.$$

Thus $2 - 2\mathrm{P} + 2\mathrm{Q} = \tfrac{5}{2} - \tfrac{1}{6}\pi^2$; and the sum of the two complementary integrals is $\tfrac{17}{8} - \tfrac{1}{6}\pi^2$. Consequently the total aggregate $\Sigma(\mathrm{V}\,dx\,dy\,dz)$ for

all eight cases combined $= 2 \times \frac{13}{8} + 6 \times \left(\frac{17}{8} - \frac{\pi^2}{6}\right) = 16 - \pi^2$. Hence the fractional part of the volume which represents the mean value required is

$$\frac{16-\pi^2}{6 \times 8}, \text{ or } \frac{1}{3} - \frac{\pi^2}{48}.$$

1859. (Proposed by J. GRIFFITHS, M.A.)—Show that the locus of the centres of equilateral hyperbolas touching the sides of a given obtuse-angled triangle is the self-conjugate circle of this triangle.

I. *Solution by* ARCHER STANLEY.

Let us first consider the system of conics inscribed in a given triangle ABC, and dividing harmonically a given rectilinear segment DE. In this system there will be but two conics which touch DE, since it is only in D or in E that two harmonic conjugates relative to D, E can coincide with one another, and only one inscribed conic can be drawn to touch DE at a given point. From this it follows that D and E are the only poles of DE, relative to the several conics of the system, which lie in DE; in other words, the *locus of all such poles is a conic* (P) *passing through* D *and* E.

If the line AD cut BC in a, then the line connecting A with a (either directly or through infinity) will represent a flattened conic (ellipse or hyperbola) included in the system; and the pole of DE, relative to it, will be the harmonic conjugate d of D relative to A, a. Hence d will be on (P). Similarly e, the harmonic conjugate of E relative to A and the intersection of AE and BC, will be another point on (P), whence it follows that BC will be the polar of A relative to (P). In a similar manner, CA is the polar of B, and AB of C; in other words ABC *is self-conjugate relative to* (P).

Allow D and E to represent the imaginary circular points at infinity, and every conic of the system will become an equilateral hyperbola, every pole of DE will be the centre of a hyperbola, and the locus (P) of this centre will be a circle relative to which ABC is a self-conjugate triangle.

II. *Solution by* S. ROBERTS, M.A.; J. DALE; E. MCCORMICK: *and others.*

Taking the given triangle as that of reference, the condition necessary in order that the general inscribed conic should be an equilateral hyperbola is

$$l^2 + m^2 + n^2 + 2mn \cos A + 2nl \cos B + 2lm \cos C = 0;$$

we have also, if (α, β, γ) be the coordinates of the *centre*,

$$l = \sin A\,(-\alpha \sin A + \beta \sin B + \gamma \sin C), \text{ \&c., \&c.};$$

hence finally the required locus is, by substitution,

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C = 0,$$

representing therefore the circle with respect to which the given triangle is self-conjugate.

[See Ferrers' *Trilinear Coordinates*, pp. 83, 42, 31.]

1857. (Proposed by Professor CAYLEY.)—If for shortness we put

$$P = x^3+y^3+z^3,\quad Q = yz^2+y^2z+zx^2+z^2x+xy^2+x^2y,\quad R = xyz,$$
$$P_0 = a^3+b^3+c^3,\quad Q_0 = bc^2+b^2c+ca^2+c^2a+ab^2+a^2b,\quad R_0 = abc\ ;$$

then (α, β, γ) being arbitrary, show that the cubic curves $\begin{vmatrix} \alpha, & \beta, & \gamma \\ P, & Q, & R \\ P_0, & Q_0, & R_0 \end{vmatrix} = 0$ pass all of them through the same nine points, lying six of them upon a conic and three of them upon a line; and find the equations of the conic and line, and the coordinates of the nine points of intersection; find also the values of $(\alpha : \beta : \gamma)$ in order that the cubic curve may break up into the conic and line.

Solution by T. COTTERILL, M.A.; PROFESSOR CREMONA; S. ROBERTS, M.A.; W. A. WHITWORTH, M.A.; J. DALE; *and others.*

The determinant vanishes, if the corresponding constituents of the second and third rows coincide, which is evidently the case for the coordinates of the six points formed from the permutations of (abc). It also vanishes, if each constituent of the second row becomes 0, or by the values

$$(x = 0,\ y+z = 0),\quad (y = 0,\ z+x = 0),\quad (z = 0,\ x+y = 0).$$

Let $\quad S = x+y+z,\quad U = x^2+y^2+z^2,\quad V = yz+zx+xy\ ;$

also $\quad S_0 = a+b+c,\quad U_0 = a^2+b^2+c^2,\quad V_0 = bc+ca+ab\ ;$

then the six points lie on the conic $UV_0 - VU_0 = 0$, and the three points lie on the line $S = 0$.

In the determinant, α, β may be changed into $\alpha+\beta$, $\beta+3\gamma$, when P and Q will become $P+Q$, $Q+3R$, which are both divisible by S.

Hence making $\alpha = -\beta = 3\gamma$, we have

$$\begin{vmatrix} 3, & -3, & 1 \\ P, & Q, & R \\ P_0, & Q_0, & R_0 \end{vmatrix} = \begin{vmatrix} P+Q, & Q+3R \\ P_0+Q_0, & Q_0+3R_0 \end{vmatrix} = SS_0\,(UV_0 - VU_0).$$

1730. (Proposed by Professor CAYLEY.) — Show that (I) the condition in order that the roots k_1, k_2, k_3 of the equation

$$\gamma k^3 + (-g-\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\tfrac{3}{2}\gamma)\,k^2 + (-g-\tfrac{3}{2}\alpha-\tfrac{1}{2}\beta+\tfrac{1}{2}\gamma)\,k - \alpha = 0 \ldots.\ (A)$$

may be connected by a relation of the form $k_3(k_1-k_2)-(k_2-k_3)=0$.. (1); and (II) the result of the elimination of a, b, c from the equations

$$a^2(b+c) = -2\alpha \ldots.\ (2),\quad b^2(c+a) = 2\beta \ldots.\ (3),\quad c^2(a+b) = -2\gamma \ldots.\ (4),$$
$$(b-c)(c-a)(a-b) = -4g \ldots.\ (5)\ ;$$

are each $4(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\,g^3 + 4(-\Sigma\alpha^3\beta + 4\Sigma\alpha^2\beta^2 - 2\Sigma\alpha^2\beta\gamma)\,g^2$
$+ (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\,g + 2(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2 = 0 \ldots\ldots\ldots\ldots (B).$

Solution by SAMUEL BILLS.

I. Here, in addition to the relation (1), we have from (A)

$$k_1+k_2+k_3 = \gamma^{-1}(g+\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta-\tfrac{3}{2}\gamma) = m \text{ (say)} \ldots\ldots\ldots\ (6),$$

$$k_2k_3 + k_3k_1 + k_1k_2 = -\gamma^{-1}(g + \tfrac{3}{2}\alpha + \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma) = n \text{ (say)} \ldots\ldots (7)$$

$$k_3^3 - mk_3^2 + nk_3 - a = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

From (1), and (6), $k_1 = \dfrac{m + (m-2)\ k_3 - k_3^2}{1 + 2k_3}$, $k_2 = \dfrac{k_3(m+1-k_3)}{1+2k_3}$.

Substituting these results in (7) we obtain

$$3k_3^4 - (2m-3)\,k_3^3 - (m^2 + 2m - 3 - 4n)\,k_3^2 - (m^2 + 2m - 4n)\,k_3 + n = 0 \ \ldots (9).$$

From $(9) - (3k_3 + m + 3)$ (8) we have

$$(m + n + 3)\,k_3^2 - (m^2 + 2m + mn - n - 3a)\,k_3 + (m+3)\,a + n = 0 \ \ldots\ldots (10).$$

Now (see Hirsch's *Algebra*, pp. 118, 119) the result of eliminating x from the equations $p + qx + rx^2 = 0$, $p_1 + q_1x + r_1x^2 + s_1x^3 = 0$, is

$$p^3s_1^2 + p^2rr_1^2 + pr^2q_1^2 + r^3p_1^2 - qr^2p_1q_1 + (q^2 - 2pr)\,(rp_1r_1 + pq_1s_1)$$
$$+ (3pqr - q^3)\,p_1s_1 - pqrq_1r_1 - p^2qr_1s_1 = 0 \ldots\ldots\ldots (C).$$

Substituting from (8) and (10) in (C), restoring the values of m and n, and arranging the result, we obtain the relation (B). It is unnecessary to give the *details* of this last step, which are easily supplied.

II. Putting $bc + ca + ab = t$, we have from $(2) - (3)$, &c., &c.,

$$(a-b)\,t = 2\,(\beta - \alpha),\ \ (b-c)\,t = 2\,(\gamma - \beta),\ \ (c-a)\,t = 2\,(\alpha - \gamma);$$

therefore by (5), $gt^3 = 2\,(\beta-\gamma)\,(\gamma-\alpha)\,(\alpha-\beta)$, which determines t;

also $a - b = 2t^{-1}\,(\beta - \alpha) = u$ (say), and $b - c = 2t^{-1}\,(\gamma - \beta) = v$ (say);

therefore $a = b + u$ and $c = b - v$; hence substituting in (2) and (4),

we have $(b+u)^2\,(2b - v) = -2\alpha$, and $(b-v)^2\,(2b+u) = -2\gamma$;

or $$2b^3 + (4u - v)\,b^2 + 2u\,(u-v)\,b - u^2v + 2\alpha = 0 \ldots\ldots\ldots\ldots (11),$$

and $$2b^3 - (4v - u)\,b^2 - 2v\,(u-v)\,b + uv^2 + 2\gamma = 0 \ldots\ldots\ldots\ldots (12).$$

From $\{(11) - (12)\} \div (u + v)$, remembering that $u + v = 2t^{-1}\,(\gamma - \alpha)$,

we have $$3b^2 + 2\,(u - v)\,b - uv - t = 0 \ldots\ldots\ldots\ldots\ldots\ldots (13).$$

Restoring the values of t, u, v, and substituting from (11) and (13) in (C), we shall obtain the relation (B) as before.

1834. (Proposed by Professor CAYLEY.)—1. It is required to find on a given cubic curve three points A, B, C, such that, writing $x = 0$, $y = 0$, $z = 0$ for the equations of the lines BC, CA, AB respectively, the cubic curve may be transformable into itself by the inverse substitution $(\alpha x^{-1}, \beta y^{-1}, \gamma z^{-1})$ in place of x, y, z respectively, α, β, γ being disposable constants.

2. In the cubic curve $ax\,(y^2 + z^2) + by\,(z^2 + x^2) + cz\,(x^2 + y^2) + 2lxyz = 0$ the inverse points (x, y, z) and (x^{-1}, y^{-1}, z^{-1}) are corresponding points (that is, the tangents at these two points meet on the curve).

I. *Solution by the* PROPOSER; S. ROBERTS, M.A.; *and others.*

Since the points A, B, C are on the curve, the equation is of the form

$$fy^2z + gz^2x + hx^2y + iyz^2 + jzx^2 + kxy^2 + 2lxyz = 0.$$

Hence this equation must be equivalent to

$$\frac{f\beta^2\gamma}{y^2z}+\frac{g\gamma^2\alpha}{z^2x}+\frac{h\alpha^2\beta}{x^2y}+\frac{i\beta\gamma^2}{yz^2}+\frac{j\gamma\alpha^2}{zx^2}+\frac{k\alpha\beta^2}{xy^2}+\frac{2l\alpha\beta\gamma}{xyz}=0,$$

$$\text{or,}\quad j\frac{\alpha}{\beta}y^2z+k\frac{\beta}{\gamma}z^2x+i\frac{\gamma}{\alpha}x^2y+h\frac{\alpha}{\gamma}yz^2+f\frac{\beta}{\alpha}zx^2+g\frac{\gamma}{\beta}xy^2+2lxyz=0,$$

which will be the case if

$$f=j\frac{\alpha}{\beta},\ g=k\frac{\beta}{\gamma},\ h=i\frac{\gamma}{\alpha},\ i=h\frac{\alpha}{\gamma},\ j=f\frac{\beta}{\alpha},\ k=g\frac{\gamma}{\beta}.$$

This implies $fgh=ijk$; and if this condition be satisfied, then $\alpha:\beta:\gamma$ can be determined, viz., we have $\alpha:\beta:\gamma=if:ij:hf$, which satisfy the remaining equations, so that the only condition is $fgh=ijk$.

Writing in the equation of the curve $x=0$, we find $fy^2z+iyz^2=0$, that is, the line $x=0$ meets the curve in the points $(x=0,\ y=0)$, $(x=0,\ z=0)$, and $(x=0,\ fy+iz=0)$. We have thus on the curve the three points

$$(x=0,\ fy+iz=0),\ (y=0,\ gz+jx=0),\ (z=0,\ hx+ky=0),$$

and in virtue of the assumed relation $fgh=ijk$, these three points lie in a line. Hence the points A, B, C must be such that BC, CA, AB respectively meet the curve in points A′, B′, C′ which three points lie in a line; that is, we have a quadrilateral whereof the six angles A, B, C, A′, B′, C′ all lie on the curve. It is well known that the opposite angles A and A′, B and B′, C and C′ must be *corresponding points*, that is, points the tangents at which meet on the curve. And conversely taking A, C any two points on the curve, A′ a corresponding point to A (any one of the four corresponding points), then AC, A′C will meet the curve in the corresponding points B′, B; and AB, A′B′ will meet on the curve in a point C′ corresponding to C, giving the inscribed quadrilateral (A, B, C, A′, B′, C′); the triangle ABC is therefore constructed.

It is to be remarked that the equation $fgh=ijk$ being satisfied, we may without any real loss of generality write $f=j,\ g=k,\ h=i$, and therefore $\alpha=\beta=\gamma$; hence changing the constants we have the theorem: the inverse points (x,y,z), (x^{-1},y^{-1},z^{-1}) are corresponding points on the curve

$$ax(y^2+z^2)+by(z^2+x^2)+cx(x^2+y^2)+2lxyz=0.$$

II. *Solution by* PROFESSOR CREMONA; J. DALE; T. COTTERILL, M.A.; *and others.*

On sait que la cubique donnée H_3 admet généralement trois systèmes de coniques tangentes en trois points (*Teoria geom. delle curve piane*, 150). Chacun de ces systèmes est formé par les poloconiques des droites du plan par rapport à une cubique C_3, dont H_3 est la Hessienne. Soit R une droite (fixe) qui coupe H_3 in $a'b'c'$; la poloconique de R (par rapport à C_3) touchera H_3 en trois points abc *correspondants* (dans le sens que M. Cayley donne à ce mot) à $a'b'c'$ ($abca'b'c'$ sont les sommets d'un quadrilatère complet). Toute conique Σ circonscrite au triangle abc est (*Teoria*, 137) la poloconique mixte de R et d'une autre droite S; et réciproquement une droite quelconque du plan détermine, avec R, une poloconique mixte, qui passe toujours par abc. Ainsi, les coniques par abc et les droites du plan constituent deux réseaux projectifs. Un point quelconque m, considéré comme commun à un faisceaux de droites (de coniques par abc) détermine un point m', commun aux coniques par abc (aux droites) correspondantes.

L'un quelconque des deux points homologues mm' est le pole de R par rapport à la conique polaire de l'autre (relative à C_3). Donc, nous avons une *transformation quadrique*, où les quatre points doubles sont les poles de R par rapport à C_3: ces poles forment un quadrangle complet dont le triangle diagonal est abc (*Teoria*, 134).

Si m est un point de H_3, la conique polaire de m est une couple de droites dont le point de croisement est correspondant à m; donc ce point est m'. C'est-à-dire que la cubique H_3 se transforme en soi-même ; et deux points homologues se *correspondent* entre eux.

On satisfait donc à la question de M. Cayley en prenant sur H_3 trois points abc où cette courbe soit touchée par une même conique K_2. Soit $\alpha yz+\beta zx+\gamma xy=0$ l'équation de K_2, et $axy^2+a'xz^2+byz^2+b'yx^2+czx^2+c'zy^2+2hxyz=0$ l'équation d'une cubique quelconque par abc. Cette cubique sera tangente à K_2 en abc si l'on a $a=m\gamma$, $a'=n\beta$, $b=n\alpha$, $b'=l\gamma$, $c=l\beta$, $c'=m\alpha$; l, m, n trois quantités arbitraires. Et la substitution $(x,y,z)=\left(\frac{\alpha}{lx},\ \frac{\beta}{my},\ \frac{\gamma}{nz}\right)$ transforme la cubique en soi-même.

En prenant $l=\alpha$, $m=\beta$, $n=\gamma$, on a la deuxième partie de la question proposée.

1871. (Proposed by Professor MANNHEIM.)—The envelope of a circle whose diameter is a chord, fixed in direction, of a given conic, is another conic whose foci are at the extremities of that diameter of the former which is conjugate to the fixed direction. Prove this, and find where the circle touches its envelope.

I. *Solution by* ARTHUR COHEN, B.A.; *and* J. H. TAYLOR, B.A.

Take as axis of x that diameter of the conic which bisects the chords of the given direction, and as axis of y the diameter perpendicular to the former. Let $2a$ be the length of the former diameter, and $2b$ the length of the conjugate diameter. Then if $2r$ be the length of any one of the chords, the coordinates of whose middle point are $(h, 0)$, we have evidently

$$\frac{r^2}{(a+h)(a-h)}=\frac{b^2}{a^2},\ \text{or}\ r^2=b^2-\frac{b^2}{a^2}h^2;$$

hence the equation to the circle on such chord as diameter is

$$(x-h)^2+y^2=r^2,\ \text{or}\ h^2\left(1+\frac{b^2}{a^2}\right)-2hx+x^2+y^2-b^2=0\ \ \ldots\ldots\ldots(1).$$

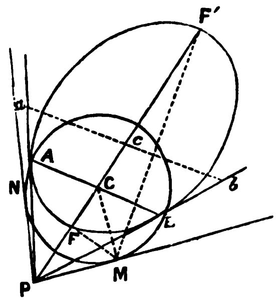
The envelope of such circles has for its equation (Salmon's *Conics*, Art. 283)

$$\left(1+\frac{b^2}{a^2}\right)(x^2+y^2-b^2)=x^2,\ \text{or}\ \frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1\ \ \ldots\ldots\ldots(2),$$

which is evidently a conic whose foci are at $(0, +a)$ and $(0, -a)$, or the extremities of the diameter conjugate to the given chords.

Comparing (1) with (2) we readily see that the abscissa x of the point where the envelope touches the circle described on the chord whose abscissa is h is given by the equation $x=h\left(1+\frac{b^2}{a^2}\right)$.

II. *Solution by* ARCHER STANLEY.



Let AB be a chord, given in direction, of the ellipse FAF'B. Let C be the middle point, and P the pole of AB, so that PC, cutting the ellipse in F and F', is the diameter conjugate to the given direction. Now, if PM and PN be the tangents from P to the circle (C) described on AB as diameter, M and N will be the points in which (C) touches its envelope. For if a parallel to AB cut the tangents PA, PB produced in a and b it is clear that the circle (c) described on ab as diameter will likewise touch PM and PN, since P will be the centre of similitude of (C) and (c), and as ab approaches AB, the circle (c) approaches to coincidence with the circle described on the chord of the ellipse which is adjacent to AB. PM and PN, therefore, are tangents at M and N to the envelope of (C).

Now, the pencil M (P, F, C, F') being manifestly harmonic, and the conjugate rays MP, MC perpendicular to one another, the latter bisects the angles formed by MF and MF'; but the tangent to the envelope at every point M thereon being equally inclined to the connectors of M with two fixed points F, F', we at once conclude that the envelope is a conic of which F and F' are the foci.

III. *Solution by the* REV. R. TOWNSEND, M.A.

The corresponding property in Geometry of Surfaces, viz.—" The envelope of a sphere whose diameter is a variable chord, fixed in direction, of a given quadric, is another quadric, of which the section of the original by the diametral plane conjugate to the fixed direction is a focal conic,"—presents no greater difficulty; for, if it be true in the particular case when the fixed direction is perpendicular to a principal plane of the surface, it is manifestly true in every case.

But in that case, the equation of the original quadric being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

if α and β be the variable coordinates of the middle point of the chord, supposed perpendicular to the plane of xy, that of the variable sphere is evidently

$$(x-\alpha)^2 + (y-\beta)^2 + z^2 = c^2\left(1 - \frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2}\right),$$

between which and the relations

$$(x-\alpha) = \frac{c^2}{a^2}\alpha, \quad (y-\beta) = \frac{c^2}{b^2}\beta,$$

obtained from it immediately by the ordinary process for the determination of envelopes, the elimination of α and β gives that of the required envelope, viz.,

$$\frac{x^2}{a^2+c^2} + \frac{y^2}{b^2+c^2} + \frac{z^2}{c^2} = 1,$$

which is therefore as above stated.

It follows, of course, from the above, conversely, that "If a variable sphere have in every position double contact with a fixed quadric, the chord of double contact being always parallel to the same axis of the surface, the extremities of its diameter parallel to the chord describe another quadric, having double contact with the original at the extremities of the axis, and passing through its focal conic in the principal plane perpendicular to the axis." The latter part of this is evident *à priori* from the known property, (see Salmon's *Geometry of Three Dimensions*, 2nd ed., Art. 138,) that every focus of a quadric is an evanescent sphere having double contact with the surface.

1861. (Proposed by C. TAYLOR, M.A.)—Prove that the difference between the sum of the sines and the sum of the cosines is greater or less than unity according as the triangle is acute or obtuse-angled.

Solution by SAMUEL ROBERTS, M.A.

The difference in question may be written

$$D = \sin A + \sin B + \sin (A+B) - \cos A - \cos B + \cos (A+B),$$

and the maxima or minima are given by

$$\sin A + \cos A = \sin B + \cos B = \sin C + \cos C, \quad \text{or} \quad \sin 2A = \sin 2B = \sin 2C,$$

with the condition $A+B+C = 180^\circ$. We have therefore the systems

$$A,\ B,\ C \text{ not } > 90^\circ \ldots \text{max. } (60^\circ, 60^\circ, 60^\circ), \text{ min. } (90^\circ, 90^\circ, 0);$$

$$A \text{ or } B \text{ or } C \text{ not } < 90^\circ \ldots \text{max. } (90^\circ, 90^\circ, 0), \text{ min. } (0, 0, 180^\circ).$$

Hence in an acute triangle, the difference in question lies between $\frac{3}{2}(\sqrt{3}-1)$ and 1, or is >1; and in an obtuse triangle, the value lies between 1 and -1, or is <1.

II. *Solution by* J. DALE; REV. J. L. KITCHIN, M.A.; *and others.*

Let A be the greatest angle; then $D > \text{or} < 1$ according as

$$2\cos\tfrac{1}{2}(A-B)\left(\cos\tfrac{1}{2}C - \sin\tfrac{1}{2}C\right) > \text{or} < 1 + \cos C - \sin C$$
$$> \text{or} < 2\cos\tfrac{1}{2}C\left(\cos\tfrac{1}{2}C - \sin\tfrac{1}{2}C\right)$$

or as $\frac{1}{2}(A-B) < \text{or} > \frac{1}{2}C$, or as $A < \text{or} > \frac{1}{2}(A+B+C) < \text{or} > 90^\circ$.

Note sur l'Intégration des Equations Différentielles Simultanées et Linéaires. *Par* E. Prouhet.

La méthode d'Ampère pour la résolution des équations

$$\left.\begin{aligned} A\frac{dy}{dx} + B\frac{dz}{dx} + Cy + Dz = E \\ A'\frac{dy}{dx} + B'\frac{dz}{dx} + C'y + D'z = E' \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots \text{(I)}$$

exige que l'on ramène ces deux équations à deux autres dont l'une ne contienne que $\frac{dy}{dx}$ et l'autre que $\frac{dz}{dx}$. On peut éviter cette opération préalable en procédant de la manière suivante.

Je suppose en premier lieu que les coefficients A, B, C, D, A', B', C', D', soient constants. J'ajoute les deux équations (I) après les avoir multipliées respectivement par 1 et par la constante θ. J'obtiens ainsi

$$\frac{d}{dx}\left\{(A+A'\theta)y+(B+B'\theta)z\right\} + (C+C'\theta)y+(D+D'\theta)z = E+E'\theta \;..\; (1).$$

Je pose $(A+A'\theta)y+(B+B'\theta)z = u \ldots. (2)$, $\frac{C+C'\theta}{A+A'\theta} = \frac{D+D'\theta}{B+B'\theta} = k .. (3)$;

et l'équation (1) se réduit à l'équation linéaire

$$\frac{du}{dx} + ku = E+E'\theta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

De la première des équations (3) on tire deux valeurs constantes (θ_1, θ_2) de l'inconnue θ. Soient u_1 et u_2 les valeurs correspondantes de u, obtenus en integrant l'équation (4). On aura

$$(A-A'\theta_1)y+(B+B'\theta_1)z = u_1, \quad (A+A'\theta_2)y+(B+B'\theta_2)z = u_2;$$

d'où l'on déduira les valeurs des fonctions inconnues y et z.

Quand les coefficients des premiers membres des équations (I) sont des fonctions de x, on procède d'une façon analogue mais en considérant θ comme une fonction de x. Dans ce cas le premier membre de l'équation (1) doit être diminué de

$$\left(\frac{dA}{dx} + \frac{dA'}{dx}\theta + A'\frac{d\theta}{dx}\right)y + \left(\frac{dB}{dx} + \frac{dB'}{dx}\theta + B'\frac{d\theta}{dx}\right)z$$

termes qu'il a fallu ajouter pour compléter la dérivée de $(A+A'\theta)y + (B+B'\theta)z$ ou de u. L'équation qui donne les valeurs de θ est alors

$$\frac{C+C'\theta - \frac{dA}{dx} - \frac{dA'}{dx}\theta - A\frac{d\theta}{dx}}{A+A'\theta} = \frac{D+D'\theta - \frac{dB}{dx} - \frac{dB'}{dx}\theta - B'\frac{d\theta}{dx}}{B+B'\theta} \ldots\ldots (5).$$

Cette équation est du premier ordre, mais non linéaire. Quand on saura l'intégrer, on aura deux valeurs de θ en attribuant à la constante arbitraire deux valeurs distinctes, et le calcul s'achèvera comme plus haut. On peut simplifier l'équation (5) en supposant que A et A' sont deux constants, ce qui est toujours permis.

La méthode précédente s'étend facilement au cas de trois équations simultanées. Nous n'examinerons que le cas où les coefficients seront constants.

Soient

$$\left.\begin{aligned} A\frac{dy}{dx} + B\frac{dz}{dx} + C\frac{dt}{dx} + Dy + Ez + Ft &= G \\ A'\frac{dy}{dx} + B'\frac{dz}{dx} + C'\frac{dt}{dx} + D'y + E'z + Ft &= G' \\ A''\frac{dy}{dx} + B''\frac{dz}{dx} + C''\frac{dt}{dx} + D''y + E''z + F''t &= G'' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (1').$$

J'ajoute ces équations respectivement multipliées par les constantes 1, θ, λ.

J'aurai $\dfrac{d}{dx}\left\{(A+A'\theta+A''\lambda)y+(B+B'\theta+B''\lambda)z+(C+C'\theta+C''\lambda)t\right\}$

$$+(D+D'\theta+D''\lambda)y+(E+E'\theta+E''\lambda)z+(F+F'\theta+F''\lambda)t$$
$$= G+G'\theta+G''\lambda\ \ldots\ldots (1').$$

Je pose $(A+A'\theta+A''\lambda)y+(B+B'\theta+B''\lambda)z+(C+C'\theta+C''\lambda)t = u\ldots (2')$,

$$\frac{D+D'\theta+D''\lambda}{A+A'\theta+A''\lambda} = \frac{E+E'\theta+E''\lambda}{B+B'\theta+B''\lambda} = \frac{F+F'\theta+F''\lambda}{C+C'\theta+C''\lambda} = k \quad \ldots\ldots\ldots (3'),$$

et l'équation (1′) se réduit à l'équation linéaire

$$\frac{du}{dx} + ku = G+G'\theta+G''\lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots (4').$$

Les équations (3′) donnent, tout calcul fait, trois valeurs de θ et trois valeurs correspondantes de λ. L'intégration de l'équation (4′) donnera trois valeurs correspondantes de u. Si l'on substitue à θ, λ, u, dans l'équation (2′) tour à tour θ_1, λ_1, u_1; θ_2, λ_2, u_2; θ_3, λ_3, u_3; on aura trois équations pour déterminer les fonctions inconnues y, z, t.

1638. (Proposed by W. K. CLIFFORD.)—Find the condition that the general equation of the third order may represent a cubic whose asymptotes form an equilateral triangle; and show that this is always the case when the curve passes through three points and their three pairs of antifoci.

Solution by the PROPOSER.

Three lines forming an equilateral triangle meet the line at infinity in a point-cubic whose Hessian is the circular points. Now let

$$(a, b, c, d)\,(x, y)^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

be the terms of the highest order in the general equation of the third degree in Cartesian coordinates; then the three lines represented by (1) are parallel to the asymptotes. Now the Hessian of (1) is $(ac-b^2)x^2+(ad-bc)xy+(bd-c^2)y^2$; and in order that this may be identical with x^2+y^2 we must have $(ad=bc,\ ac-b^2=bd-c^2)$, which are the conditions required.

To prove the proposition, let A, B, C be the three points, and P, Q the circular points at infinity. Let the equation to the three lines PA, PB, PC be $U=0$, and to the three lines QA, QB, QC, $V=0$. Then the nine intersections of the cubics U, V are the three points and their three pairs of antifoci. Any other cubic through those intersections may be represented by $U=kV$. Let U′, V′ be the terms of highest order in U, V; then

$U'-kV'$ will be the terms of highest order in $U-kV$. But U', V' must be perfect cubes, representing the circular points; say x^3, y^3. Then $U'-kV'$ is x^3-ky^3. But the Hessian of x^3-ky^3 is $-kxy$. That is, every cubic represented by $U=kV$ meets the line at infinity in a point-cubic whose Hessian is the circular points. Or, which is the same thing, the asymptotes of every such cubic form an equilateral triangle.

There is no difficulty in finding the conditions when the equation is given in a homogeneous form. We substitute for z, from the equation of the line at infinity, in the cubic and in any circle; let the former substitution give (1), and the latter, $Ax^2+2Bxy+Cy^2=0$; then the conditions are

$$\frac{ac-b^2}{A}=\frac{ad-bc}{2B}=\frac{bd-c^2}{C}.$$

1825. (Proposed by H. R. GREER, B.A.)—Prove that, in space, the locus of a point such that, if perpendiculars be drawn from it to the faces of a tetrahedron, their feet shall lie in a plane, is the surface

$$\frac{A}{x}+\frac{B}{y}+\frac{C}{z}+\frac{D}{w}=0,$$

A, B, C, D representing the areas of the faces, and x, y, z, w the perpendiculars drawn on them from any point.

Solution by W. ALLEN WHITWORTH, M.A.

Let $$l\alpha+m\beta+n\gamma+r\delta=0 \quad\text{.........................}\quad \text{(i.)}$$

be the equation to the plane containing the feet of the perpendiculars from (x, y, z, w) on the faces A, B, C, D; then, if cos AB, cos BC, &c. denote the *external* angles between the planes A and B, B and C, &c., so that, by projection,

$$A+B\cos AB+C\cos AC+D\cos AD=0 \text{ (and similar equations) .... (ii.)},$$

the equations to the perpendicular from (x, y, z, w) on A may be written

$$\frac{\alpha-x}{1}=\frac{\beta-y}{\cos AB}=\frac{\gamma-z}{\cos AC}=\frac{\delta-w}{\cos AD}=\rho,$$

and since this line meets the plane (i.) on the plane $\alpha=0$, we get

$$m(y-x\cos AB)+n(z-x\cos AC)+r(w-x\cos AD)=0.$$

We get three similar equations from considering the other perpendiculars. Therefore, eliminating $l:m:n:r$, we get

$$\begin{vmatrix} 0\ , & y-x\cos AB, & z-x\cos AC, & w-x\cos AD \\ x-y\cos BA, & 0\ , & z-y\cos BC, & w-y\cos BD \\ x-z\cos CA, & y-z\cos CB, & 0\ , & w-z\cos CD \\ x-w\cos DA, & y-w\cos DB, & z-w\cos DC, & 0 \end{vmatrix}=0,$$

which, in virtue of (ii.), becomes

$$\begin{vmatrix} Ax+By+Cz+Dw, & \&c. & \\ Ax+By+Cz+Dw, & \&c. & \left\{\begin{array}{l}\text{the other columns}\\ \text{as before.}\end{array}\right. \\ Ax+By+Cz+Dw, & \&c. & \\ Ax+By+Cz+Dw, & \&c. & \end{vmatrix}=0,$$

or $$\begin{vmatrix} \frac{1}{x}, & \frac{1}{x}-\frac{1}{y}\cos AB, & \frac{1}{x}-\frac{1}{z}\cos AC, & \frac{1}{x}-\frac{1}{w}\cos AD \\ \frac{1}{y}, & 0\quad , & \&c. & \&c. \\ \frac{1}{z}, & \frac{1}{z}-\frac{1}{y}\cos BC, & \&c. & \&c. \\ \frac{1}{w}, & \frac{1}{w}-\frac{1}{y}\cos BD, & \&c. & \&c. \end{vmatrix} = 0,$$

or again, in virtue of (ii.),

$$\begin{vmatrix} \frac{A}{x}+\frac{B}{y}+\frac{C}{z}+\frac{D}{w}, & \frac{A}{x}+\frac{B}{y}+\frac{C}{z}+\frac{D}{w}, & \frac{A}{x}+\frac{B}{y}+\frac{C}{z}+\frac{D}{w}, & \frac{A}{x}+\frac{B}{y}+\frac{C}{z}+\frac{D}{w} \\ \frac{1}{y} \quad , & 0 \quad , & \frac{1}{y}-\frac{1}{z}\cos BC\ , & \frac{1}{y}-\frac{1}{w}\cos BD \\ \frac{1}{z} \quad , & \frac{1}{z}-\frac{1}{y}\cos BC, & 0 \quad , & \frac{1}{z}-\frac{1}{w}\cos CD \\ \frac{1}{w} \quad , & \frac{1}{w}-\frac{1}{y}\cos BD, & \frac{1}{w}-\frac{1}{z}\cos CD\ , & 0 \end{vmatrix} = 0,$$

or $$\left(\frac{A}{x}+\frac{B}{y}+\frac{C}{z}+\frac{D}{w}\right)\begin{vmatrix} 0 \quad , & \frac{1}{z}\cos BC, & \frac{1}{w}\cos BD \\ \frac{1}{y}\cos BC\ , & 0 \quad , & \frac{1}{w}\cos CD \\ \frac{1}{y}\cos BD, & \frac{1}{z}\cos CD, & 0 \end{vmatrix} = 0,$$

or $$\frac{1}{y}\cdot\frac{1}{z}\cdot\frac{1}{w}\left(\frac{A}{x}+\frac{B}{y}+\frac{C}{z}+\frac{D}{w}\right) = 0;$$

hence $\frac{A}{x}+\frac{B}{y}+\frac{C}{z}+\frac{D}{w} = 0$ represents the locus required.

1867. (Proposed by W. H. LAVERTY.)—From a point O_1 on a conic, $(n-1)$ lines are drawn to points $O_2, O_3 \ldots O_n$ on the conic; $O_1O_3, O_1O_4 \ldots O_1O_n$ being inclined at angles $\alpha_3, \alpha_4 \ldots \alpha_n$ to O_1O_2; find the product of $O_1O_2 . O_1O_3 \ldots O_1O_n$, (1) in the general case, (2) when the conic becomes a circle, (3) when $O_1O_2O_3 \ldots O_n$ is a regular polygon in the circle.

Solution by S. ROBERTS, M.A.; REV. J. L. KITCHIN, M.A.; *the* PROPOSER; *and others.*

We may take O_1 as origin of rectangular coordinates. Then

$(A\cos^2\alpha + B\sin\alpha\cos\alpha + C\sin^2\alpha)\,\rho^2 + (D\cos\alpha + E\sin\alpha)\,\rho = 0$ gives

$$(1) \ldots \rho_1\rho_2 \ldots \rho_{n-1} = (-)^{n-1}\frac{D}{A} \cdot \frac{D\cos\alpha_3 + E\sin\alpha_3}{A\cos^2\alpha_3 + B\sin\alpha_3\cos\alpha_3 + C\sin^2\alpha_3} \cdots\cdots$$

$$\cdots\cdots \frac{D\cos\alpha_n + E\sin\alpha_n}{A\cos^2\alpha_n + B\sin\alpha_n\cos\alpha_n + C\sin^2\alpha_n};$$

$$(2) \ldots \rho_1\rho_2 \ldots \rho_{n-1} = (-)^{n-1} D \, . \, A^{1-n} \, . \, (D\cos\alpha_3 + E\sin\alpha_3) \ldots (D\cos\alpha_n + \&c.);$$

(3).. since the side of a regular polygon of n sides is $2R\sin\frac{\pi}{n}$, we have

$$\rho_1\rho_2 \ldots \rho_{n-1} = (-)^{n-1}\, 2R\sin\frac{\pi}{n} \, . \, 2R\sin\frac{2\pi}{n} \ldots . \, 2R\sin\frac{(n-1)\pi}{n}.$$

But by Euler's formula we have

$$\sin\theta\sin\left(\theta + \frac{\pi}{n}\right) \ldots \sin\left\{\theta + \frac{(n-1)\pi}{n}\right\} = 2^{1-n} . \sin n\theta, \text{ whence}$$

$$\sin\frac{\pi}{n} \, . \sin\frac{2\pi}{n} \ldots \sin\frac{(n-1)\pi}{n} \, . = 2^{1-n} \, . \, \frac{\sin n\theta}{\sin\left\{\theta + \frac{(n-1)\pi}{n}\right\}}\left(\theta = \frac{\pi}{n}\right) = 2^{1-n} n;$$

therefore $\rho_1\rho_2 \ldots \rho_{n-1} = nR^{n-1}$, a result immediately deducible also from Cotes' theorem, for we have $\rho_1\rho_2 \ldots \rho_{n-1} = \frac{x^n - R^n}{x - R}\,[x = R] = nR^{n-1}$.

1699. (Proposed by W. GODWARD.)—Let O_1, O_2, O_3, be the centres of the escribed circles touching the sides BC, CA, AB, respectively, of the triangle ABC; and P_1, P_2, P_3 the feet of the perpendiculars from the vertices A, B, C on those sides; prove that O_1P_1, O_2P_2, O_3P_3 intersect in a point the sum of the trilinear coordinates of which is $\frac{R+r}{R-r}r$.

Solution by W. H. LAVERTY; J. DALE; E. FITZGERALD; *the* PROPOSER; *and others.*

Writing, for shortness' sake, (l, m, n) for $(\cos A, \cos B, \cos C)$ the trilinear equations of O_1P_1, O_2P_2, O_3P_3 are readily found to be

$$m\beta - n\gamma = (n-m)\,\alpha, \quad n\gamma - l\alpha = (l-n)\,\beta, \quad l\alpha - m\beta = (m-l)\,\gamma;$$

and since $l\,(\text{eq. } O_1P_1) + m\,(\text{eq. } O_2P_2) + n\,(\text{eq. } O_3P_3) \equiv 0$, these three lines meet in a point whose coordinates, found from any two of the equations, are

$$\frac{\alpha}{m+n-l} = \frac{\beta}{n+l-m} = \frac{\gamma}{l+m-n} = \frac{\Sigma(\alpha)}{\Sigma(\cos A)} = \frac{2\Delta}{\Sigma(a) - \Sigma(a\cos A)}$$

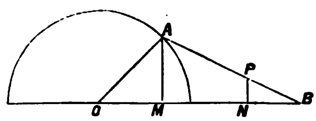
$$= \frac{2\Delta}{\frac{2\Delta}{r} - \frac{2\Delta}{R}} = \frac{Rr}{R-r};$$

$\therefore\ \Sigma(a) = \dfrac{r}{R-r}\,\Sigma(R\cos A) = \dfrac{R+r}{R-r}\,.\,r.$ (McDowell's *Exercises*. Prop. 96).

[It is clear that the concurrence of the lines in nowise depends on the values of l, m, n (see also the Note at the end of the Solution of Quest. 1726): and the sum of the coordinates of the centre of homology of the triangles $O_1O_2O_3$, $P_1P_2P_3$, for *any* values of l, m, n, is $\Sigma(a) = \dfrac{(l+m+n)\,\Delta}{ls_1+ms_2+ns_3}$. If, for example, $l:m:n = a:b:c$, the result agrees with that given in the Note at the end of the Solution of Quest. 1616 (*Reprint*, Vol. III., p. 31.]

1703. (Proposed by C. BICKERDIKE.)—Given the length of the connecting rod of a horizontal steam engine, and the length of the stroke; find the locus of a given point in the connecting rod during one revolution of the crank.

Solution by J. DALE; E. FITZGERALD; E. MCCORMICK; *and many others.*



Let OA = half the length of the stroke $= r$; AB (the connecting rod) $= a$; P any given point in AB; and PB $= b$. Then the question is to find the locus of P when A moves round the circle (O). Taking O as origin, OB as axis of x, and putting ON $= x$, PN $= y$, $\angle$AON $= \theta$, $\angle$PBN $= \phi$, we have $r\cos\theta + a\cos\phi = x + y\cot\phi$; also $\sin\phi = \dfrac{y}{b}$,

$$\sin\theta = \frac{a}{\gamma}\sin\phi = \frac{ay}{br},\quad \cos\phi = \frac{1}{b}(b^2-y^2)^{\frac{1}{2}},\quad \cos\theta = \frac{1}{br}(b^2r^2-a^2y^2)^{\frac{1}{2}};$$

hence, substituting these values in the above equation, we get, for the equation of the required locus,

$$(a-b)\,(b^2-y^2)^{\frac{1}{2}} + (b^2r^2-a^2y^2)^{\frac{1}{2}} = bx,$$

whence we see that the locus is a curve of the fourth order.

If $r=a$, the locus becomes $\dfrac{x^2}{(2a-b)^2} + \dfrac{y^2}{b^2} = 1$, which represents an ellipse whose semi-axes are $2a-b$ and b.

[When $r=a$, the *complete* locus consists of a semi-ellipse and a semicircle, having a common diameter perpendicular to OB through O; these *two* equations of the *second* degree being respectively obtained from Mr. DALE's equation of the *fourth* degree by taking the radicals therein of the *same* or of *different* signs. This will also be seen to be in conformity with the geometrical circumstances of the motion, the range of B being $2r$.]

On some extensions of the fundamental proposition in M. Chasles' Theory of Characteristics. By W. K. Clifford.

I mean by the "fundamental proposition" the following, viz.:—

"If a variable system of two points on a right line be so related that when the second point is taken arbitrarily the first has a positions, and when the first point is taken arbitrarily the second has b positions; then there are $a+b$ points on the right line at which the system of two points coalesces into one point."

This principle has been admirably extended by Dr. Salmon to the case of two dimensions, thus:—"If a variable system of two points in a plane be so related that when the second point is taken arbitrarily the first has a positions, and when the first point is taken arbitrarily the second has b positions, and that p pairs of points, each constituting a position of the system, may be found upon an arbitrary right line; then there are $a+b+p$ points in the plane at which the system of two points coalesces into one point."

The principle admits of further extension in two directions. First, we may consider a system of more than two points; and secondly, we may consider the system as subject to a less number of relations than is sufficient to determine a single point. We are thus led to the following propositions:—

If a variable system of n points in space be so related that when all but the first point are taken arbitrarily the first point is determined to lie on a surface of order a, and when all but the second point are taken arbitrarily the second point is determined to lie on a surface of order b, and so on; then there are points in space at which the system of n points coalesces into one point, and the locus of such points is a surface of order $a+b+\ldots.$

If a variable system of n points in space be so related that when all but the first point are taken arbitrarily the first point is determined to lie on a curve of order a, and when all but the second point are taken arbitrarily the second point is determined to lie on a curve of order b, and so on, and that when all but the first two points are taken arbitrarily there are on an arbitrary right line p pairs of points each constituting a position of the first two points, and that $q, r, \ldots.$ are the corresponding numbers for the other pairs of points of the system; then there are points in space at which the system of n points coalesces into one point, and the locus of such points is a curve of order $a+b+\ldots..+p+q+r+\ldots.$

It is not worth while to state the analogous propositions for Geometry of one and two dimensions, or the correlative propositions for lines and planes. I go on to exemplify the application of these propositions.

Let us begin with Mr. Thomson's cubic. (Quest. 1545; *Reprint*, Vol. II., p. 57). A conic is inscribed in a triangle so that the normals at the points of contact meet in a point; it is required to find the locus of this point. Consider now a variable system of three points, subject only to this condition; that if perpendiculars be drawn from them respectively to the three sides of a triangle, a conic may be drawn touching the sides of the triangle at the feet of those perpendiculars. Then if we take two of the points arbitrarily, we determine two of the points of contact of the inscribed conic; that is, we determine the conic itself uniquely, and therefore the third point of contact; and the normal at this point is therefore the locus of the third point of the system. That is to say, we have a variable system of three points so related that when any two of the points are taken arbitrarily, the locus of the third is a straight line; consequently there are points in the plane at which the system of three points coalesces into one point (that is, where the

three normals meet in a point), and the locus of such points is a curve of order $(1+1+1=)\,3$.

To complicate the question, let us suppose that a conic is drawn to touch three given *conics*, so that the normals at the points of contact may meet in a point. Here, as before, we take our variable system of three points, one on each of the three normals. Take two of the points arbitrarily; from each of these we can draw four normals to the corresponding conic. Pairing these together, we have 16 pairs of points of contact. Now when we have given two tangents and their points of contact, the number of conics of the system which can be drawn to touch a given conic is 4. By determining two points of our variable system we have, therefore, determined 64 conics; on the third *given* conic these determine 64 points of contact, and the normals through these may be held to constitute a curve of the 64th order. Thus we have a variable system of three points so related that when any two of them are taken arbitrarily, the third is determined to lie on a curve of the 64th order; consequently the locus of those points at which the system coalesces into one point, or the three normals meet in a point, is a curve of the order $(3\times 64=)\,192$. More generally, if we substitute for the variable conic a curve of order m, class n, and deficiency D, or say a curve of *species* (m, n, D), and for the three fixed conics $\frac{1}{2}(m+n-D+2)$ curves of orders $m_1, m_2, \ldots$ and of classes $n_1, n_2, \ldots$, then the corresponding locus will be of the order $3\phi(m, n, D).(m_1+n_1)(m_2+n_2)\ldots$, where $\phi(m, n, D)$ is the number of curves of species (m, n, D) which can be drawn through $(m+n-D+1)$ points, or touching $(m+n-D+1)$ lines.

For another example, let us find the locus of those points the feet of the perpendiculars from which to four lines or planes in space are coplanar. In both these cases the locus comes out primarily of the fourth order; but the plane at infinity is evidently a part of the locus, the remainder of which is thus of the third order. In both cases the envelope of the plane through the feet of the perpendiculars is of the fourth class, and touches the plane at infinity. I conjecture that the imaginary circle is a curve of contact.

If a conicoid be drawn to touch five straight lines, so that the normal planes at the points of contact meet in a point, the locus is of this point of the tenth order. And so on *ad libitum*.

1854. (Proposed by Chief Justice COCKLE, F.R.S.)—Solve the differential equation $\frac{dy}{dx}+by^2=ax^2$; or, differential added to multiple of square of dependent variable equal to multiple of square of independent variable.

Solution by the PROPOSER.

The following discussion includes all cases of the equation

$$\frac{dy}{dx}+by^2=ax^m \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

with the exception of that in which $m=1$. But in this excepted case the Solution may (by means of the investigations given at pp. 81, 82 of Hymers's

Differential Equations, 1839) be expressed by a definite integral. Hence it may be said that the equation (1) is in all cases whatever solvable either by definite or indefinite integrals.

Instead of (1) let us take as our starting point the equation

$$\frac{d^2y}{dx^2} + C^2\left\{(2n+1)x\right\}^{\frac{m-4n}{2n+1}}y = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

to which (1) may always by appropriate transformations be reduced, and in which C, m, and n are arbitrary constants which are not supposed to be subject to any other conditions than that neither C nor $2n+1$ shall vanish. This being so, let

$$x = \frac{t^{2n+1}}{2n+1}, \quad \therefore \frac{dx}{dt} = t^{2n}, \quad \therefore \frac{dt}{dx}\cdot\frac{d^2x}{dt^2} = \frac{2n}{t} \ldots\ldots\ldots (3, 4, 5),$$

then, changing the independent variable from x to t, (2) becomes

$$\frac{d^2y}{dt^2} - \frac{2n}{t}\cdot\frac{dy}{dt} + C^2t^my = 0 \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Now change the dependent variable, and let

$$y = t^nv, \quad \therefore \frac{dy}{dt} = t^n\frac{dv}{dt} + nt^{n-1}v \ldots\ldots\ldots\ldots\ldots (7, 8),$$

and
$$\frac{d^2y}{dt^2} = t^n\frac{d^2v}{dt^2} + 2nt^{n-1}\frac{dv}{dt} + n(n-1)t^{n-2}v \ldots\ldots\ldots\ldots (9).$$

Then, after substitution and reduction, (6) becomes

$$\frac{d^2v}{dt^2} + \left\{C^2t^m - (n^2+n)\frac{1}{t^2}\right\}v = 0 \ldots\ldots\ldots\ldots\ldots (10).$$

Let $m=0$, then, as we know (Hymers, ibid. pp. 83—85), the equation

$$\frac{d^2v}{dt^2} + \left\{C^2 - (n^2+n)\frac{1}{t^2}\right\}v = 0 \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

has for its solution $v = \beta t^{-n}\left(\frac{d^n}{d\xi^n}\right)_{\xi=C^2}\left\{\frac{\cos(t\sqrt{\xi}+\alpha)}{\sqrt{\xi}}\right\} \ldots\ldots\ldots (12)$

when n is an integer: while the same equation (11) has for its solution

$$v = \beta t^{n+1}\int_{-C}^{C}(\xi^2 - C^2)^n\cos(t\xi+\alpha)\,d\xi \ldots\ldots\ldots\ldots\ldots (13)$$

when, in (10), n is a fraction; and in either solution α and β are arbitrary constants.

To solve the equation proposed in the Question, I put it under the successive forms

$$\frac{dy}{dx} + y^2 = ax^2, \text{ and } \frac{d^2y}{dx^2} + ax^2y = 0;$$

and then (remembering that the m in (2), which is an arbitrary quantity not necessarily identical with the m in (1), is to vanish) I put

$$2 = -\frac{4n}{2n+1}, \quad \text{or} \quad 4n+2 = -4n, \quad \text{or} \quad n = -\frac{1}{4} \ldots\ldots\ldots (14),$$

and hence obtain a reduced equation which, by properly assigning a, may be expressed by

$$\frac{d^2v}{dt^2} + \left\{1 - \left(\frac{1}{16} - \frac{1}{4}\right)\frac{1}{t^2}\right\} v = 0 \quad \text{.............. (15)},$$

whereof the solution is (compare Hymers, ibid.)

$$v = \beta t^{\frac{1}{2}} \int_{-1}^{1} \frac{\cos(t\xi + a)\, d\xi}{\sqrt[4]{(\xi^2 - 1)}} \quad \text{.................. (16)},$$

and whereon that of the proposed equation depends. For any value of m, the index of the dexter of (1), we have, in place of (14), the conditions

$$m = -\frac{4n}{2n+1}, \quad \text{or} \quad 2mn + m = -4n, \quad \text{or} \quad n = -\frac{m}{2m+4} \quad \text{...... (17)},$$

and so we may solve the general case. But the particular cases have certain relations one to the other which may be important as suggesting relations between the definite integrals which enter into them. These relations I hope to enter upon to some extent in discussing another Question proposed by me.

If, instead of (1), we had taken as our starting point the equation

$$\frac{d^2y}{dx^2} + \left[C^2 \left\{(2n+1)\, x\right\}^{\frac{m-4n}{2n+1}} + \frac{n^2+n}{(2n+1)^2} \cdot \frac{1}{x^2}\right] y = 0 \quad \text{........ (18)},$$

we should have been led, by means of the conditions (3, 4, 5), to the transformed equation

$$\frac{d^2y}{dt^2} - \frac{2n}{t} \cdot \frac{dy}{dt} + \left\{C^2 t^{m-4n} + (n^2+n)\, t^{-2(2n+1)}\right\} t^{4n} y = 0 \quad \text{...... (19)},$$

and thence, by means of the conditions (7, 8) and (9), to the transformation

$$\frac{d^2v}{dt^2} + C^2 t^m v = 0 \quad \text{....................... (20)},$$

the solution of which is thus seen to depend upon that of (18). Now, in order that the process hereinbefore used may be applicable to (18), we must have

$$m - 4n = 0, \quad \text{and} \quad \frac{n^2+n}{(2n+1)^2} = -(N^2+N) \quad \text{.......... (21, 22)},$$

the latter of which conditions is equivalent to

$$\frac{1}{4}\left\{1 - \frac{1}{(2n+1)^2}\right\} = -(N^2+N) \quad \text{.......... (23)},$$

whence $\quad \dfrac{1}{2n+1} = \pm(2N+1) \quad \text{or} \quad 2N+1 = \dfrac{\pm 1}{2n+1}$ .... (24, 25),

and hence, again, $\quad N = \dfrac{1}{2}\left(\dfrac{-2n-1 \pm 1}{2n+1}\right)$ .... (26),

or, in virtue of (21), $\quad N = \dfrac{1}{2}\left(\dfrac{-\frac{1}{2}m - 1 \pm 1}{\frac{1}{2}m + 1}\right) = \dfrac{-m-2 \pm 2}{2m+4}$ .... (27),

whence $\quad m = \dfrac{-4N - 2 \pm 2}{2N+1}$ .... (28).

Now, as we have seen, if N be an integer, (20), being reducible to a form solvable by indefinite integrals, is itself so solvable. Accordingly, on sub-

stituting for N in (28) any integers positive or negative, we are led to the ordinary forms of RICCATI, which however are here comprised in a single formula of solution. If (27) gives rise to a fractional value of N, then (20) is solvable by means of definite integrals. I have already noticed the isolated case in which the formulæ fail.

1838. (Proposed by M. W. CROFTON, B.A.)—Two steamers are continually running between a port and two given points, subtending a given angle at the port, and each of which is just visible from it; find the chance of the steamers being visible to one another at any particular instant.

Solution by the PROPOSER.

The question may be otherwise enunciated thus :—

Two equal lines CA, CB, of length r include an angle θ; to find the chance that if two points P, Q, be taken at random, one on each line, their distance PQ shall be less than r.

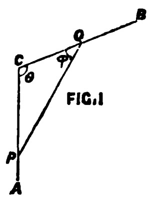


FIG. 1

1. When $\theta > \frac{1}{2}\pi$. Let $CP = x$, $CQ = y$, $\angle CQP = \phi$; P, Q being two points such that $PQ = r$; then $x = r \sin\phi \operatorname{cosec}\theta$, $y = r\sin(\phi+\theta)\operatorname{cosec}\theta$; and if F be the measure of the favourable cases, it is easy to see that

$$F = \int_0^r x\,dy = \frac{r^2}{\sin^2\theta}\int_{\pi-\theta}^{0} \sin\phi\cos(\phi+\theta)\,d\phi = \frac{r^2(\pi-\theta)}{2\sin\theta}.$$

Now the measure (W) of the whole number of cases is r^2; hence the required probability is

$$p = \frac{F}{W} = \frac{\pi-\theta}{2\sin\theta}.$$

2. When $\theta > \frac{1}{3}\pi$ and $< \frac{1}{2}\pi$. Draw from A (Fig. 2) a line $AV = r$; then the value of F will be

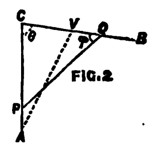


FIG. 2

$$F = \int_{CV}^{r} x\,dy + CV\,.\,AC$$

$$= \frac{r^2}{\sin^2\theta}\int_{\theta}^{\pi-2\theta} \sin\phi\cos(\phi+\theta)\,d\phi + 2r^2\cos\theta;$$

whence we find $$p = \frac{3\theta-\pi}{2\sin\theta} + 2\cos\theta.$$

3. When $\theta < \frac{1}{3}\pi$, it is obvious that $p = 1$.

NOTE.—If $F(\theta)$ be the function expressing the probability in the first case, $f(\theta)$ in the second, we find they are related by the remarkable equation

$$F(\theta) + F(\pi-\theta) = f(\theta) + f(\pi-\theta).$$

From this and other examples (see solution to Quest. 1321, *Reprint*, Vol. IV., p. 86, NOTE) it would appear that the functions expressing the probabilities in problems which have different cases, are connected by very singular and simple relations, which have not yet been studied.

1858. (Proposed by R. BALL, M.A.)—If in any symmetric function of the differences of the roots of an equation, each root a_k be changed into $\dfrac{1}{(a_k - x)}$, show that the result, when cleared of fractions, will be a covariant.

Solution by SAMUEL ROBERTS, M.A.

It is evident that the result of making such substitution, and clearing of fractions, is a symmetric function of differences of roots and differences of x and of one or more of the roots, and each root enters term by term the same number of times. The result, therefore, is a covariant.

1883. (Proposed by R. TUCKER, M.A.)—Draw a straight line parallel to a given straight line to cut a given semicircle so that the trapezoid formed by the chord, the diameter, and the perpendiculars on the diameter from the points of section may be given or a maximum.

Solution by J. H. TAYLOR, B.A.

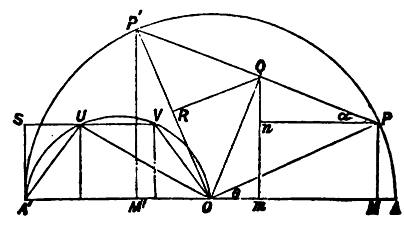
Let PP′ be parallel to the given line, and the angles OQP, QmA, PnQ right angles; then $\angle$ QPn $= \alpha$ (a known $\angle$). Let $\angle$ POA $= \theta$, and OA $= a$;

then $Pn = QP \cos\alpha$

$= a \cos\alpha \cos(\alpha + \theta)$,

and $Qm = OQ \cos\alpha$

$= a \cos\alpha \sin(\alpha + \theta)$;



hence area of trapezoid $= 2\,Pn \,.\, Qm = a^2 \cos^2\alpha \sin 2(\alpha + \theta)$.

When therefore the area is constant ($= c^2$), θ can be determined from the equation

$$a^2 \cos^2\alpha \sin 2(\alpha + \theta) = c^2.$$

The area is a *maximum* when $2(\alpha + \theta) = \frac{1}{2}\pi$, that is when $\theta = \frac{1}{4}\pi - \alpha$, and then OP, OP′ are at right angles to each other.

[*Otherwise:* The area of the right-angled triangle OQP′ is clearly *given* or a *maximum* at the same time with that of the trapezoid MP′; moreover the line OQ is given in position, being perpendicular to the given line to which PP′ is to be parallel; hence we readily obtain a *geometrical* solution by determining the point Q in OQ so that the perpendicular QR from Q on OP′ may be *given* or a *maximum*. This may be done by placing a line A′S perpendicular to A′O and equal to the *given* perpendicular QR, then from S drawing SUV parallel to A′O to meet a semicircle on A′O in U and V, and lastly making OQ equal to OU or OV. The triangle (and therefore the trapezoid) will be a *maximum* when U coincides with V, and then OUA′, OQP′, OQP are isosceles right-angled triangles.

The foregoing Solutions will hold good in all cases if, when PP′ cuts the diameter, the part of the trapezoid *below* AA′ be considered *negative*, in accordance with the general theory of signs.]

1888. (Proposed by E. DE JONQUIÈRES.)—(1.) Amongst the conics which have three-pointic contact with a cubic at a given point, there are, in general, three which have a three-pointic contact elsewhere and a fourth passes through the points of contact of these three with the cubic. The number of such conics is reduced to one, when the cubic has a cusp.

(2.) Amongst the conics which have four-pointic contact with a cubic at a given point there are three which touch the cubic elsewhere. There is but one such conic when the cubic has a node, and none when it has a cusp. [From the *Nouvelles Annales de Mathématiques.*]

I. *Solution by* PROFESSOR HIRST.

1. Any three distinct or coincident points ABC being taken on a cubic, and considered as fundamental points, that cubic becomes transformed, by quadric inversion, into another cubic, which likewise passes through ABC and has, in general, the same singularities as the primitive curve (see *Proceedings of the Royal Society* for March, 1865). At the same time every conic through ABC which has three pointic contact elsewhere with the primitive cubic becomes transformed into a stationary tangent of the inverse cubic. M. DE JONQUIERE's first theorem, therefore, is the inverse of the well known one, according to which a cubic has, in general, three stationary tangents, whose points of contact are collinear.

There is but one such tangent, of course, when the cubic has a cusp.

2. The second theorem also corresponds by quadric inversion to the following very simple one:—From the point wherein a cubic is intersected by any one of its tangents three other tangents can, in general, be drawn to the curve; only one, however, can be so drawn if the cubic has a node, and none if it has a cusp.

The theorem, however, may be demonstrated directly with equal readiness. If a, b, c, d be any four points on the cubic, it is well known that every conic through them intersects the curve again in two points which are collinear with a fixed point o on the curve, and conversely every line through o cuts the cubic in two points which lie with a, b, c, d on a conic. Now in

general four of these lines, and hence four of the conics, touch the cubic. When a, b, c, d are coincident, however, one of these four conics breaks up into two lines coincident with the tangent, and the three conics alluded to in the theorem alone remain. The modification which the theorem suffers when the cubic has a node or a cusp is manifest.

II. *Solution by* W. K. Clifford.

1. Let A be the given point on the cubic, and let F be any point of inflexion, or flex. Join AF, and let AF meet the curve again in B. Then a conic may be drawn having three-pointic contact with the cubic at the points A and B. For, consider these three cubic curves :—(a) the cubic itself: (b) the line ABF taken three times over; (c) a conic having three-pointic contact at A and touching the cubic at B, together with the tangent at the flex F. The cubic (c) passes through eight out of the nine points of intersection of the cubics (a) and (b); consequently, by the theorem known as the involution of cubics, it passes through the ninth point. That is to say, a conic having three-pointic contact at A, and touching the cubic at B, will necessarily have three-pointic contact at B.

By joining the point A, therefore, to the *nine* flexes F, we shall obtain *nine* points B, and therefore nine conics fulfilling the required conditions; but only three of these points B will be real when the point A is real.

It remains to show that a conic having three-pointic contact at A passes through the three real points B. Let F_1, F_2, F_3 be the three real flexes, which are known to be in one straight line; and let B_1, B_2, B_3 be the corresponding points B. Draw a conic U having three-pointic contact at A and passing through B_1, B_2. Then consider these three cubic curves;—(a) the cubic itself; (b) the straight lines AB_1F_1, AB_2F_2, AB_3F_3; (c) the conic U and the line $F_1F_2F_3$. The cubic (c) passes through eight out of the nine intersections of the cubics (a) and (b); consequently it passes also through the ninth. That is to say, the conic U passes through the point B_3.

A cusped cubic has only one flex; in this case, therefore, the number of conics is reduced to one.

2. Let A be the given point. By Cotterill's Theorem (which again is a particular case of the involution of cubics), if a conic have four-pointic contact with the cubic at A, its remaining chord of intersection with the cubic will pass through a fixed point M on the curve. Now the tangent at A, taken twice over, may be regarded as a conic having four-pointic contact at A; whence it appears that the point M is the second tangential of A. The number of conics of the system which touch the cubic at some other point is therefore the number of tangents that can be drawn from M to the curve; that is, four in general, two when the cubic has a node, and one when it has a cusp. But in this number there is always included that conic which is made up of the tangent at A taken twice over; and this is not a proper solution.

Theorem concerning Five Points on a Circle.
By John Griffiths, M.A.

If we are given five points on the circumference of a circle of radius r (say), I propose to show that the centres of the five equilateral hyperbolas

which pass through them, taken four and four together, will lie on the circumference of another circle, whose radius is $\frac{1}{2}r$.

Let us take any two diameters of the given circle at right angles to each other as axes of coordinates; then the equation of the equilateral hyperbola which passes through the four points whose angular ordinates are $\alpha, \beta, \gamma, \delta$ can be put under the form

$$\left\{x \cos \tfrac{1}{2}(\alpha+\beta) + y \sin \tfrac{1}{2}(\alpha+\beta) - r \cos \tfrac{1}{2}(\alpha-\beta)\right\} \times$$
$$\left\{x \cos \tfrac{1}{2}(\gamma+\delta) + y \sin \tfrac{1}{2}(\gamma+\delta) - r \cos \tfrac{1}{2}(\gamma-\delta)\right\} = \lambda(x^2+y^2-r^2),$$

where $\cos \frac{1}{2}(\alpha+\beta) \cos \frac{1}{2}(\gamma+\delta) + \sin \frac{1}{2}(\alpha+\beta) \sin \frac{1}{2}(\gamma+\delta) = 2\lambda.$

But this evidently reduces to

$$\cos \tfrac{1}{2}(\alpha+\beta+\gamma+\delta) \,.\, (x^2-y^2) + 2 \sin \tfrac{1}{2}(\alpha+\beta+\gamma+\delta) \,.\, xy$$
$$-2r\left\{\cos \tfrac{1}{2}(\gamma+\delta) \cos \tfrac{1}{2}(\alpha-\beta) + \cos \tfrac{1}{2}(\alpha+\beta) \cos \tfrac{1}{2}(\gamma-\delta)\right\} x$$
$$-2r\left\{\sin \tfrac{1}{2}(\gamma+\delta) \cos \tfrac{1}{2}(\alpha-\beta) + \sin \tfrac{1}{2}(\alpha+\beta) \cos \tfrac{1}{2}(\gamma-\delta)\right\} y + \mathrm{F} = 0,$$

F being the constant term.

From this the coordinates of the centre are easily found to be

$$x = \tfrac{1}{2}r(\cos\alpha + \cos\beta + \cos\gamma + \cos\delta), \; y = \tfrac{1}{2}r(\sin\alpha + \sin\beta + \sin\gamma + \sin\delta).$$

If, then, the angular ordinates of the five given points be $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$, the coordinates of the centres of the corresponding hyperbolas will be

$$\left.\begin{array}{l} x_1 = \tfrac{1}{2}r\,\Sigma \cos\phi - \tfrac{1}{2}r \cos\phi_5 \\ y_1 = \tfrac{1}{2}r\,\Sigma \sin\phi - \tfrac{1}{2}r \sin\phi_5 \end{array}\right\} \quad \left.\begin{array}{l} x_2 = \tfrac{1}{2}r\,\Sigma \cos\phi - \tfrac{1}{2}r \cos\phi_1 \\ y_2 = \tfrac{1}{2}r\,\Sigma \sin\phi - \tfrac{1}{2}r \sin\phi_1 \end{array}\right\}, \;\&\mathrm{c.};$$

whence it is easily seen that the five centres lie on the circle

$$(x - \tfrac{1}{2}r\,\Sigma\cos\phi)^2 + (y - \tfrac{1}{2}r\,\Sigma\sin\phi)^2 = \tfrac{1}{4}r^2,$$

which proves the theorem.

NOTE.—Let A, B, C, D, E denote any five points on a circle; then it follows from the above that the consecutive intersections of the nine-point circles of the triangles ABC, BCD, CDE, DEA, EAB, lie on another circle, whose radius is one half that of the first.

If, however, the given points be taken on *any conic section*, the curve which passes through the five intersections in question will not in general be a circle.

ADDITION TO THE NOTE ON THE PROBLEMS IN REGARD TO A CONIC DEFINED BY FIVE CONDITIONS OF INTERSECTION. BY PROFESSOR CAYLEY.

Since writing the Note in question, I have found that a solution of Problem 7 has been given by M. De Jonquières in the paper "Du Contact des Courbes Planes, &c.," *Nouvelles Annales de Mathématiques*, Vol. III. (1864),

pp. 218—222: viz., the number of conics which touch a curve of the order n in five distinct points is stated to be

$$\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}(n^5+15n^4-55n^3-495n^2+1584n+15).$$

There are given also the following results; the number of conics which pass through two given points and touch a curve of the order n in three distinct points is

$$\frac{n(n-1)(n-2)}{2}(n^3+6n^2-19n-12),$$

and the number of conics which pass through a given point and touch a curve of the order n in four distinct points is

$$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}(n^4+10n^3-37n^2-118n+282).$$

These formulæ are given without demonstration, and with an expression of doubt as regards their exactness—("elles sont exactes, je crois"); they apply, of course, to a curve of the order n without singularities; but assuming them to be accurate, the means exist for adapting them to the case of a curve with singularities.

[There is also a paper on the same subject in the *Annales* for January, 1866 (pp. 17—20), from the Editor's *Note* to which we have introduced a correction ($+15$ instead of -35) in the formula given above.]

1876. (Proposed by R. Ball, M.A.)—If three of the roots of the equation $(a, b, c, d, e)(x, 1)^4 = 0$ be in arithmetical progression, show that

$$55296H^3J-2304aH^2I^2-16632a^2HIJ+625a^3I^3-9261a^3J^2=0,$$

where $H = ac-b^2$, $I = ae-4bd+3c^2$, $J = ace+2bcd-ad^2-b^2e-c^3$.

I. *Solution by* Professor Cayley.

Write $(a, b, c, d, e)(x, 1)^4 = a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$; then putting for a moment $\beta+\gamma+\delta = p$, $\beta\gamma+\beta\delta+\gamma\delta = q$, $\beta\gamma\delta = r$, and forming the equation

$$(\beta+\gamma-2\delta)(\beta+\delta-2\gamma)(\gamma+\delta-2\beta) = 0,$$

this is easily reduced to $\quad -2p^3+9pq-27r = 0.$

But we have $a(x^3-px^2+qx-r)(x-\alpha) = (a, b, c, d, e)(x, 1)^4$, and hence

$$p = -\frac{4b}{a}-\alpha,\ q = \frac{6c}{a}+\frac{4b}{a}\alpha+\alpha^2,\ r = -\frac{4d}{a}-\frac{6c}{a}\alpha-\frac{4b}{a}\alpha^2-\alpha^3.$$

Substituting these values of p, q, r, the foregoing equation becomes, after all reductions,

$$(20a^3,\ 20a^2b,\ -16ab^2+36a^2c,\ 128b^3-216abc+108a^2d)(\alpha, 1)^3 = 0,$$

and from this and the equation $(a, b, c, d, e)(\alpha, 1)^4 = 0$, eliminating α, we should find the condition for three roots in arithmetical progression. But it appears from the theory of invariants that the result of the elimination may be obtained by writing $b = 0$, and expressing the result so obtained in terms of a, H, I, J. Hence, writing in the two equations $b = 0$, the first equation contains the factor $4a^2$, and throwing this out, the equations become

$$5a\alpha^3 + 27c\alpha + 27d = 0,\ a\alpha^4 + 6c\alpha^2 + 4d\alpha + e = 0\,;$$

or multiplying the first by α and reducing by means of the second, the two equations become

$$5a\alpha^3 + 27c\alpha + 27d = 0,\quad 3c\alpha^2 - 7d\alpha + 5e = 0.$$

The result is of the degree 5 in the coefficients, but in order to avoid fractions in the final result it is proper to multiply it by a^4; it then becomes

$$625\,a^6e^3 - 4050\,a^5c^2e^2 + 6561\,a^4c^4e - 1890\,a^5ced^2 + 13122\,a^4c^3d^2 + 9261\,a^5d^4 = 0.$$

But writing as above $b = 0$, we have

$$a = a,\ c = \frac{\text{H}}{a},\ e = \frac{\text{I}}{a} - \frac{3\text{H}^2}{a^3},\ d^2 = -\frac{\text{J}}{a} + \frac{\text{HI}}{a^2} - \frac{4\text{H}^3}{a^4}\,;$$

and substituting these values, the result is found to contain the terms $\frac{\text{IH}^4}{a}$, $\frac{\text{H}^6}{a^3}$ with coefficients which vanish; viz., the coefficient of the first of these terms is $+16875 + 24300 + 6561 + 7560 + 18792 - 74088,\ = 0\,;$

and the coefficient of the second of the two terms is

$$-16875 - 36450 - 19683 - 75168 + 148176,\ = 0.$$

The remaining terms give

$$\left.\begin{array}{lll} +625 & = & +625\,a^3\text{I}^3 \\ -5625-4050-1890+9261 & = & -2304\,a\text{H}^2\text{I}^2 \\ +1890-18522 & = & -16632\,a^2\text{HIJ} \\ -18792+74088 & = & +55296\,\text{H}^3\text{J} \\ +9261 & = & +9261\,a^3\text{J}^2 \end{array}\right\} = 0,$$

which is the required result; a more convenient form of writing it is

$$(55296\,\text{J},\ -768\,\text{I}^2,\ -5544\,\text{IJ},\ 625\,\text{I}^3 + 9261\,\text{J}^2)(\text{H}, a)^3 = 0.$$

REMARK.—If I and J denote as above the two invariants of the form $\text{U} = (a, b, c, d, e)(x, 1)^4$, and if we now use H to denote the Hessian of the form, viz.,

$$\text{H} = \left\{ac - b^2,\ \tfrac{1}{2}(ad - bc),\ \tfrac{1}{6}(ae + 2bd - 3c^2),\ \tfrac{1}{2}(be - cd),\ ce - d^2\right\}(x, 1)^4,$$

then it appears by the theory of invariants that the equation of the twelfth order

$$(55296\,\text{J},\ -768\,\text{I}^2,\ -5544\,\text{IJ},\ 625\,\text{I}^3 + 9261\,\text{J}^2)(\text{H}, \text{U})^3 = 0,$$

is such that each of its roots forms with some three of the roots of the equation $\text{U} = 0$ a harmonic progression; viz., if the three roots are β, γ, δ, then we have

$$\frac{2}{x-\gamma} = \frac{1}{x-\beta} + \frac{1}{x-\delta},\ \text{or}\ x = \frac{2\beta\delta - (\beta+\delta)\gamma}{\beta + \delta - 2\gamma}\,;$$

so that the roots of the equation of the twelfth order are the twelve values of the last mentioned function of three roots.

II. *Solution by* S. BILLS.

Assume $x = y - \frac{b}{a}$; then, taking for H, I the above mentioned values, and putting $G = 2b^3 - 3abc + a^2d$, the equation in question becomes

$$y^4 + 6\frac{H}{a^2}y^2 + 4\frac{G}{a^3}y + \frac{a^2I - 3H^2}{a^4} = 0 \ldots\ldots\ldots\ldots (1),$$

Now if three of the roots of the given equation be in arithmetical progression, it is obvious that three of the roots of (1) will be so likewise, since the respective roots differ by a constant.

But the roots of (1), supposing three of them to be in arithmetical progression, are evidently of the form $p+q,\ p,\ p-q,\ -3p$. Taking the sum of the products of every two, of every three, and the product of all four, we shall have

$$6p^2 + q^2 = -\frac{6H}{a^2},\quad p(4p^2 - q^2) = \frac{2G}{a^3},\quad 3p^2(p^2 - q^2) = \frac{3H^2 - a^2I}{a^4} \ldots (2,3,4).$$

From (2), $q^2 = -\frac{6H}{a^2} - 6p^2$; substituting this in (3) and (4), we have

$$5p^3 + \frac{3H}{a^2}p - \frac{G}{a^3} = 0,\quad 21p^4 + \frac{18H}{a^2}p^2 + \frac{a^2I - 3H^2}{a^4} = 0 \ldots\ldots\ldots (5,6).$$

But since p is a root of (1), we have

$$p^4 + 6\frac{H}{a^2}p^2 + 4\frac{G}{a^3}p + \frac{a^2I - 3H^2}{a^4} = 0 \ldots\ldots\ldots\ldots (7).$$

From (6) − 21 (7) we obtain, after a little reduction,

$$27a^2Hp^2 + 21aGp - 5(3H^2 - a^2I) = 0 \ldots\ldots\ldots\ldots (8).$$

Eliminating p between (5) and (8), as shown in my Solution of Quest. 1730 (*Reprint*, Vol. V., p. 38) and substituting for G^2 its value $-4H^3 + a^2IH - a^3J$, we obtain

$$55296\,H^3J - 2304\,aH^2I^2 - 16632\,a^2HIJ + 625\,a^3I^3 - 9261\,a^3J^2 = 0.$$

1472. (Proposed by the EDITOR.)—1. Find two positive rational numbers such that if from each of them, and also from the sum of their squares, their product be subtracted, the three remainders may be rational square numbers.

2. Find two positive rational numbers such that if from each of them, and also from the square root of the sum of their squares, their product be subtracted, the three remainders may be rational square numbers.

Solution by S. BILLS; W. HOPPS; S. WATSON; *and others.*

1. Let x and y be the required numbers; then we must have

$$x - xy = \text{a square number} = p^2x^2 \text{ suppose} \ldots\ldots\ldots\ldots (1),$$

$$y - xy = \text{a square number} = q^2y^2 \text{ suppose} \ldots\ldots\ldots\ldots (2),$$

$$x^2 + y^2 - xy = \text{a square number} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From (1) and (2), $x = \dfrac{q^2-1}{p^2q^2-1}$, $y = \dfrac{p^2-1}{p^2q^2-1}$;

and these values being substituted in (3), it becomes (putting $\square$ to denote a rational square number)

$$(p^2-1)^2+(q^2-1)^2-(p^2-1)(q^2-1) = \square,$$

or, by putting $q+1 = p-1$,

$$(p-3)^2+(p+1)^2-(p+1)(p-3) = \square,$$

therefore $\quad p^2-2p+13 = \square = (p-z)^2$, suppose;

whence $p = \dfrac{z^2-13}{2(z-1)}$, where z may be taken at pleasure.

If $z=2$, we have $p = -\dfrac{9}{2}$, $q = -\dfrac{13}{2}$; and two numbers satisfying the conditions are $x = \dfrac{60}{1243}$, $y = \dfrac{28}{1243}$; the three squares being

$$\left(\frac{270}{1243}\right)^2, \ \left(\frac{182}{1243}\right)^2, \ \left(\frac{52}{1243}\right)^2.$$

2. Let x and y denote the numbers, and assume $y=1-x$ and $x^2+y^2=v^2$, then the first two conditions are satisfied, and it remains to find

$$x^2+(1-x)^2 = v^2, \quad \text{and} \quad v-xy=\square \ldots\ldots (4, 5).$$

From (4), $v^2-x^2 = (1-x)^2$, and to satisfy this condition assume

$$(v+x) = (p-1)(1-x), \quad \text{and} \quad (p-1)(v-x) = (1+x),$$

then $\quad x = \dfrac{p^2-2p^2}{p^2-2}, \quad y = \dfrac{2p-2}{p^2-2}, \quad v = \dfrac{p^2-2p+2}{p^2-2}.$

Substituting in (5), we shall have to make

$$p^4-4p^3+6p^2-4=\square=(p^2-2p+1)^2, \text{ suppose};$$

then we find $p = \frac{5}{4}$, which does not give positive values for *both* x and y; therefore assume $p = q+\frac{5}{4}$, then by substituting in the preceding expression it becomes

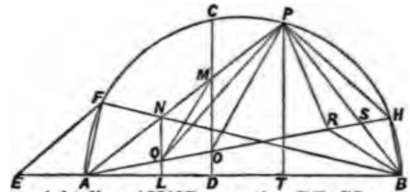
$$q^4+q^3+\frac{3}{8}q^2+\frac{65}{16}q+\frac{1}{256} = \square = \left(q^2-\frac{65}{2}q-\frac{1}{16}\right)^2, \text{ say};$$

then $q = \dfrac{4223}{264}$, and $p = \dfrac{4553}{264}$; whence we find

$$x = \frac{18325825}{20590417}, \ y = \frac{2264592}{20590417}, \ v = \frac{18465217}{20590417}, \ v-xy = \left(\frac{18403967}{20590417}\right)^2.$$

1326. (Proposed by J. Conwill.)—Find a point in the arc of a semicircle such that, if it be joined with the ends of the diameter, the quadrilateral contained between the joining lines, the diameter, and a given perpendicular to the diameter, may be a maximum.

Solution by the PROPOSER; W. HOPPS; *and others.*



Let AB be the diameter of the semicircle, and CD the given perpendicular thereto, which we will suppose to be nearer to A than to B. Produce BA to E (Euc. VI., 29) so that $AE \,.\, EB = 2AD^2$; draw the chord $AF = AE$, and bisect the angle BAF by the straight line ANMP, meeting BF, CD, and the arc BC, in N, M, P respectively; then P will be the required point, and PMDB the maximum quadrilateral.

Draw NL and PT perpendicular to AB; and, H being *any other* point in the arc BC, let AH meet NL, CD, PB in Q, O, S respectively. Join PO, PH, MQ; through B draw BR parallel to PH, meeting AS in R, and join PR.

Then, because AP bisects the angle BAF, it is easy to show that the tangent at P is parallel to BF; therefore the point R will be in HQ, or in HQ produced, according as the point H is in the arc BP, or in the arc PC.

Now $AL = AF = AE$, and $2AT = BA + AF = BE$ (McDowell's *Exercises*, Prop. 77),

therefore $\quad 2AD^2 = BE \,.\, AL = 2AT \,.\, AL,$ whence $AL : AD = AD : AT,$

therefore $\quad AQ : AO = AM : AP;$

hence PO is parallel to MQ, $\triangle PQO = PMO$, and $\triangle PQH = PMOH$.

But when H is in the arc BP, $\triangle PBH \; (= PRH) < PQH$,

therefore $\quad PMOS > \triangle SBH$, and therefore $PMDB > HODB$.

The proof applies, with a slight modification, when the point H is in the arc PC; moreover it is obvious that the required point is in the arc BC; hence the quadrilateral PMDB is a *maximum*.

[The foregoing proof (that the quadrilateral PMDB is a maximum) will apply just as well to *any segment* ACB of a circle, provided AP be drawn so as to satisfy the two following conditions:—(i.) that the tangent at P shall be parallel to BF, which will *always* be the case when AP bisects the angle BAF; (ii.) that PO shall be parallel to MQ, which will require that AD shall be a mean proportional between AT and AL, or, since $BA + AF = 2AT$,

that $\quad (BA + AF) \,.\, AL = 2AD^2 \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (A).$

When ACB is a semicircle, $AL = AF$; hence, in this case, making $AE = AF = AL$, (A) becomes

$$BE \,.\, EA = 2AD^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (B),$$

which, together with the condition (i.), is the construction in the above solution.

When however the segment ACB is *not* a semicircle, AL is not equal to AF, and the position of AP is not given by the construction in (B).

The conditions (A) and (B) of construction may be readily obtained by the ordinary analytical processes:—thus, let $AB = a$, $AD = c$, $\angle BPA = \alpha$, $\angle BAP = \theta$; then we have

$$\text{Area of PMDB} = \tfrac{1}{2}\left\{a^2 \operatorname{cosec}\alpha \sin\theta \sin(\theta + \alpha) - c^2 \tan\theta\right\},$$

hence the quadrilateral PMDB will be a minimum when

$$c^2 \sec^2\theta = a^2 \operatorname{cosec}\alpha \sin(2\theta + \alpha), \;\text{ or }\; AM^2 = BA \,.\, AF = PA \,.\, AN \;\ldots\; (C),$$

that is to say, when AM and AD are mean proportionals between AP and AN, AT and AL, respectively; or, what amounts to the same thing, when

$$(BA+AF) . AL = 2AD^2, \text{ which is the condition (A).}$$

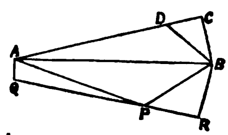
When the segment ACB is a semicircle, $\alpha = \frac{1}{2}\pi$ and (C) becomes

$a^2 \cos 2\theta \cos^2 \theta = c^2$, or $(a \cos 2\theta)(a + a \cos 2\theta) = 2c^2$, or $BE . EA = 2AD^2$.]

1320. (From the *Lusus Seniles* of the Rev. JOHN SAMPSON.)—

Quid faciam, docti, carum visurus amicum,
  Quem late extensâ degere valle juvat?
Hujus ab æde domus tredecim mea millia distat,
  Quadrigis rapidis attamen ire queam
Cauponam versus distantem millia bis sex;
  Millia cauponâ quinque et amicus abest.
Quadrigis hora sex millia curritur unâ,
  Quatuor interea millia vado pedes.
Quam longe utemur quadrigis, dicite tandem,
  Tempore quo *minimo* conficiamus iter?

Solution by the Rev. W. MASON; F. COWLEY; *and others.*



The inn (C) is at a point 5 miles distant from the friend's house (B), and since $AB^2 (= 13^2 = 12^2 + 5^2) = AC^2 + CB^2$, the direction of the foot-path (CB) to this house from the inn is at right angles to the highway (AC).

Let D be the point to leave the highway, and put $CD = x$; then the time (in hours) along ADB is

$$t = \frac{12-x}{6} + \frac{\sqrt{(x^2+25)}}{4}; \;\therefore\; -\frac{x}{6} + \frac{\sqrt{(x^2+25)}}{4} = t-2 = t',$$

and t' is obviously positive: hence putting $T = 12t'$, we have

$$9(x^2+25) = (T+2x)^2, \text{ or } 5x^2 - 4Tx + (225 - T^2) = 0.$$

The *least* value of T is found from the equation

$$(4T)^2 = 4 \times 5 \times (225 - T^2), \;\therefore\; T = 5\sqrt{5}, \text{ and } x = \tfrac{2}{5}T = 2\sqrt{5}.$$

Hence $t = 2 + \frac{5}{12}\sqrt{5}$, and time in $AD = 2 - \frac{1}{3}\sqrt{5} = 1\frac{1}{4}$ hours, nearly.

[The following is a more general form of the problem:—

Qiven two points A, B, two numbers m, n, and any line QR, to find a point P in QR so that $\dfrac{AP}{m} + \dfrac{PB}{n}$ may be a minimum.

Draw AQ, BR perpendicular to the given line, and put $AQ = a$, $BR = b$, $QR = c$, $\angle APQ = \phi$, $\angle BPR = \psi$; then, when QR is a *straight* line, we have

$a \cot \phi + b \cot \psi = c$, and $na \operatorname{cosec} \phi + mb \operatorname{cosec} \psi =$ minimum ...... (1, 2).

Differentiating (1) and (2), the condition for a minimum is found to be

$$m \cos\psi = n \cos\phi, \text{ or } \cos\phi : \cos\psi = m : n \ldots\ldots\ldots\ldots (3);$$

and the position of P is completely determined by (1) and (3).

When A and B are on opposite sides of QR, we have the case of a ray of light passing from one medium into another of different density; and the result (3) shows that the time of passing from A in one medium to B in the other is a minimum when the sines of the angles of incidence and refraction, made with a normal at P, are proportional to the velocities (m, n) of propagation of light in the two media.

When $m = n$, then by (3) the angle APQ = BPR, and we have the solution of a well-known case of the general problem; that, namely, in which AP + PB is a minimum.

In the particular case proposed in the question, QR takes the position of the highway ADC; m and n are the rates by coach and on foot, respectively; and (3) gives

$$\cos \mathrm{BDC} = \frac{n}{m} = \frac{2}{3} = \frac{\mathrm{DC}}{\mathrm{DB}}; \therefore \mathrm{AD} = \text{distance by coach} = 12 - 5\cot \mathrm{D} = 12 - 2\sqrt{5}.]$$

1851. (Proposed by A. RENSHAW.)—Given four points in a plane; show that the equation which determines the coefficient of xy, in any conic passing through the four points, so that the circumscribing rectangle may be a maximum or a minimum, is of the third order.

Solution by J. DALE; E. McCORMICK; *the* PROPOSER; *and others.*

Let $ay^2 + bxy + cx^2 + dy + ex + 1 = 0$ be the equation to a conic; then if four points in it be given, we can determine the coefficients a, c, d, e, which may therefore be regarded as known. Referring the conic to the centre as origin, and the principal axes, the above equation becomes

$$\left\{a + c \pm \sqrt{(a-c)^2 + b^2}\right\} y^2 + \left\{a + c \mp \sqrt{(a-c)^2 + b^2}\right\} x^2$$
$$= 2\left(\frac{ae^2 + cd^2 - bde}{4ac - b^2} - 1\right).$$

Hence we find that the circumscribing rectangle depends upon the function

$$\frac{ae^2 + cd^2 - bde}{(4ac - b^2)^{\frac{3}{2}}} - \frac{1}{(4ac - b^2)^{\frac{1}{2}}}.$$

Differentiating this with respect to b, and equating the result to 0, we get

$$b^3 - 2deb^2 + (3ae + 3cd^2 - 4ac)\,b - 4acd = 0,$$

and from this equation of the third degree b is to be determined.

[The question is nearly identical with the problem, To draw the least ellipse through four given points, which has been discussed by EULER in the *Petersburgh Transactions*, (Vol. IX., p. 132,) and by Messrs. FENWICK and HEARN in the *Mathematician*. (Vol. II., pp. 233, 316.)]

ON THE PROBLEMS IN REGARD TO A CONIC DEFINED BY FIVE CONDITIONS OF INTERSECTION. BY PROFESSOR CAYLEY.

There has been recently published in the *Comptes Rendus* (tom. 62, pp. 177—183, Jan. 1866) an extract of a memoir "Additions to the Theory of Conics," by M. H. G. ZEUTHEN (of Copenhagen). The extract gives the solutions of fourteen problems, with a brief indication of the method employed for obtaining them. Of these problems, four relate to intersections at given points, the remaining ten are included among the twenty-seven problems enumerated in my *Note* on this subject in the January Number of the *Educational Times* (*Reprint*, Vol. V., p. 25); but two of these ten are the problems 25 and 26 which are in my *Note* stated to have been solved; there are, consequently, of the twenty-seven problems, in all twelve which

No. of Prob.	1,	8,	10,	12,	14,	17,	19,	21,	23,	25,	26,	27
Zeuthen's No.	—,	14,	13,	11,	8,	3,	12,	7,	2,	6,	1,	—

are solved: viz., these are where it is to be observed that ZEUTHEN'S solutions apply to the case of a curve of a given order with given numbers of double points and cusps. The problems 25 and 26 had been previously solved only in the case of a curve without singularities. As to Prob. 27, the solution mentioned in my former *Note* is in fact applicable to the general case. The solution for Prob. 1 may also be extended to this general case, viz., for a curve of the order m with δ double points and κ cusps the required number is $= m(12m-27)-24\delta-27\kappa$; or, if n be the class, then this number is $= 12n-15m+9\kappa$: so that all the twelve problems are solved in the general case.

The results obtained by M. DE JONQUIERES, as stated in my *Note* in the March Number (*Reprint*, Vol. V., p. 57), seem to be all of them erroneous. In fact, for the number of conics passing through two given points and touching a curve of the order m in three distinct points (which is a particular case of Prob. 23), ZEUTHEN'S formula applied to a curve without singularities gives this

$$= \tfrac{1}{8}m\,(m-2)\,(m^4+5m^3-17m^2-49m+108)$$

instead of the value $\tfrac{1}{8}m\,(m-1)\,(m-2)\,(m^3+6m^2-19m-12)$ which is

$$= \tfrac{1}{8}m\,(m-2)\,(m^4+5m^3-25m^2+7m+12);$$

and I have by my own investigation verified ZEUTHEN'S Number. So for the number of conics through a given point and touching a curve of the order m in four distinct points (which is a particular case of Prob. 17), ZEUTHEN'S formula applied to a curve without singularities gives this

$$= \tfrac{1}{24}m\,(m-2)\,(m-3)\,(m^5+9m^4-15m^3-225m^2+140m+1050)$$

instead of the value

$$\tfrac{1}{24}\,m\,(m-1)\,(m-2)\,(m-3)\,(m^4+10m^3-37m^2-118m+282)$$

which is

$$= \tfrac{1}{24}\,m\,(m-2)\,(m-3)\,(m^5+9m^4-47m^3-81m^2+400m-282),$$

and it may I think be inferred that the expression obtained for the number of conics which touch a given curve in five distinct points (Prob. 7), containing as it does the factor $(m-1)$, is also erroneous.

I have obtained for Prob. 2 a solution which I believe to be accurate; viz., the number of the conics (4, 1), (that is, the conics which have with a given curve a 5-pointic intersection and also a 2-pointic intersection, or ordinary contact), is

$$=10n^2+10nm-20m^2-130n+140m+10\kappa(m+n-9)-4[(n-3)\kappa+(m-3)\iota]$$

where ι (the number of inflexions) is $=3n-3m+\kappa$, but I prefer to retain the foregoing form, without effecting the substitution.

1495. (Proposed by HUGH GODFRAY, M.A.)—Show that $\frac{1}{8}n(n-1)(n-2)$ points can always be so arranged in a plane that they shall be situated by eights in $\frac{1}{24}n(n-1)(n-2)(n-3)$ circles.

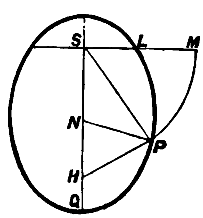
Solution by SAMUEL ROBERTS, M.A.

If n points be taken four together, we shall have $\frac{1}{24}n(n-1)(n-2)(n-3)$ sets. Each set, considered as a quadrangle, determines a circle passing through the intersections of the diagonals and the opposite sides. To each triangle of the quadrangle, correspond $(n-3)$ such circles. The number of triangles being $\frac{1}{6}n(n-1)(n-2)$, the whole series of circles can be formed into the same number of sets of $(n-3)$. Since four triangles belong to a quadrangle, each circle will reappear four times in the sets; and two circles will not occur together more than once. All this is of course on the assumption that there is no special limitation of the points, and all the circles are different.

Suppose that the $(n-3)$ circles of each set pass through the same two points. This supposition is permissible, since no two circles occur together twice. It follows that all the points of intersection in question are of the $(n-3)$rd order, and each circle has upon it 8 such points, while the total number of them is $\frac{1}{8}n(n-1)(n-2)$.

1641. (Proposed by E. McCORMICK.)—An ellipse is placed with its major axis vertical; find, geometrically, the straight line of quickest descent from the upper focus to the curve.

Solution by G. C. DE MORGAN, M.A.; *and the* REV. J. L. KITCHIN, M.A.



Let S, H be the foci. Draw SLM perpendicular to the major axis, making LM equal to the semi-parameter SL. Describe a circle with centre S and radius SM, meeting the ellipse in P; then SP is the straight line required.

For, making the angle NPS = NSP, we have, if e be the eccentricity of the ellipse,

$$\cos \mathrm{NSP} = \frac{1}{e}\left(1-\frac{\mathrm{SL}}{\mathrm{SP}}\right)=\frac{1}{2e},$$

whence it follows that $\mathrm{SN}=\mathrm{NP}=e\,.\,\mathrm{SP}$. Hence

since $SH = e(SP + HP)$, we have $SN : NH = SP : PH$, therefore NP is the normal at P, and the proposition follows at once. [For a circle drawn round N as centre with radius NP would pass through S and *touch* the ellipse at P.]

If $e < \frac{1}{2}$, the circle never meets the ellipse, but in that case, Q being the lower end of the major axis, the circle on SQ as diameter lies wholly within the ellipse, and SQ is the required line.

1798. (Proposed by Professor SYLVESTER.)—

(1.) Let $f(x) = 1 + x + \dfrac{x^2}{1.2} + \dfrac{x^3}{1.2.3} \dots + \dfrac{x^n}{1.2\dots n}$,

prove that $f(x)$ cannot have two real roots.

(2.) Let $\phi(x) = 1 + \nu x + \dfrac{\nu(\nu+1)}{1.2}x^2 + \dots + \dfrac{(\nu+1)\dots(\nu+n-1)}{1.2\dots n}x^n$,

if $\nu > 0$ or $< -n$, prove that $\phi(x)$ cannot have two real roots.

(3.) Deduce (1) from (2).

I. *Solution by the* PROPOSER.

If $F(x) = c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \dots + c_n$, and we write

$$a_0 = c_0, \quad a_1 = \frac{c_1}{\mu}, \quad a_2 = \frac{1.2.c_2}{\mu(\mu+1)}, \quad \dots\dots \quad a_n = \frac{1.2.3\dots n}{\mu(\mu+1)\dots(\mu+n-1)}.c_n,$$

it has been shown (see the late Mr. PURKISS' paper in the *Messenger of Mathematics*, Vol. III., pp. 129—142) that the number of imaginary roots in Fx cannot fall short of the number of changes of sign in the progression

$$a_0^2, \quad a_1^2 - a_0a_2, \quad a_2^2 - a_1a_3, \quad \dots\dots \quad a_{n-1}^2 - a_{n-2}a_n, \quad a_n^2,$$

provided that μ is not intermediate between 0 and $-n$.

In (2), changing x into $\dfrac{1}{x}$, we have $x^n + \nu x^{n-1} + \dfrac{\nu(\nu+1)}{1.2}x^{n-2} + \dots = 0$. Hence, making $\mu = \nu$, we have $a_0 = 1$, $a_1 = 1$, ... $a_n = 1$; and the criterion series becomes 1, 0, 0, ... 0, 1; which, since any zero is free to be taken + or − indifferently, is equivalent to the series $+ - + - \dots \pm +$, whence it is evident that all, or all but one, of the roots are imaginary.

We may derive (1) from (2) by taking $\nu = \infty$ and writing x for νx, or at once from the general theorem by taking $\mu = \infty$.

If we wished to apply Newton's rule (as given by Newton) to the equation $fx = 0$, *i. e.*, $x^n + nx^{n-1} + n(n-1)x^{n-2} + \dots = 0$; his criterion series, omitting square factors, being what my series becomes when μ is taken equal to n, would be

$$1; \quad 1^2 - 2; \quad 2^2 - 1.3; \quad 3^2 - 2.4; \quad \dots\dots \quad (n-1)^2 - n(n-2); \quad 1;$$

i.e. $1; \quad -1; \quad 1; \quad 1; \quad \dots \quad 1; \quad 1;$

and as there are but *two* changes of sign in the above, we could infer the certain existence of not more than *one pair* of imaginary roots, which well illustrates the importance of the arbitrary parameter (limited) which I have imported into the theorem.

II. *Solution by the* REV. J. BLISSARD.

This Question proposed in its most general form would be as follows:—

Let $F_n x = \frac{a_0 x^n}{1.2\ldots n} + \frac{a_1 x^{n-1}}{1.2\ldots(n-1)} + \ldots + a_{n-1}x + a_n$.

Required the conditions necessary to be fulfilled in order that $F^n x = 0$ must (n even) have all its roots unreal, or (n odd) all unreal but one.

Let $F_1 x = a_0 x + a_1$ (a_0 positive), $F_2 x = \frac{a_0 x^2}{1.2} + a_1 x + a_2$, &c.

It is evident that $F_2 x, F_3 x \ldots F_n x$ are all obtained by successive Integration from $F_1 x$, the constants $a_2, a_3 \ldots a_n$ being respectively added. Also, since $F_{n-1} x$ is derived from $F_n x$ by Differentiation, $F_n x = 0$ must have at least as many unreal roots as $F_{n-1} x = 0$. Hence if n is odd and all the roots of $F_{n-1} x = 0$ are unreal, $F_n x = 0$ can only have one real root; and if n is even and $F_{n-1} x = 0$ has only one real root. $F_n x = 0$ must have at least $n-2$ unreal roots. It is now required to determine the conditions under which $F_n x = 0$ (n even) *must* have all its roots unreal. For this purpose it is evidently necessary that a_n should be positive and > 0, since otherwise two roots of $F_n x = 0$ must be real. Now let a_n receive all values from 0 to $+\infty$; then $F_n x = 0$ must begin ($a_n = 0$) with having two real roots (one being zero) and end ($a_n = \infty$) with having all its roots unreal, for in this case the equation $F_n x = 0$ is reduced to $x^n = -\infty$, all the roots of which must be unreal. Hence, as a_n passes from 0 to ∞, two roots of $F_n x = 0$ must pass from reality through equality to unreality. There must consequently be a positive value of a_n which renders two roots of $F_n x = 0$ real and equal, and $F_{n-1} x = 0$ must have one root the same as these. It follows that if x be eliminated between $F_n x = 0$ and $F_{n-1} x = 0$, the result of this elimination (denoted by C_n) when arranged according to the powers of a_n must give the conditions of reality, equality, and unreality of two roots of $F_n x = 0$, according as that result, viz. C_n, is negative, zero, or positive. Hence, if C_2, C_4, &c. be the results respectively of the elimination of x between $F_1 x = 0$ and $F_2 x = 0$, between $F_3 x = 0$ and $F_4 x = 0$, and so on, these functions, viz. C_2, C_4, &c. will be $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ in number, as n is even or odd; and if they are all positive, then $F_n x = 0$ must (n even) have all its roots unreal, and (n odd) all unreal but one.

If r is the real root in $F_{n-1} x = 0$, then $F_{n-1} r = 0$; and if $F_n r$ is positive, all the roots of $F_n x = 0$ (n even) must be unreal. For if a_n be of such a value that two roots of $F_n x = 0$ are equal, r must be that root, and $F_n r = 0$. Hence, as a_n varies from 0 to ∞, $F_n r$ will be negative, zero, or positive, according as $F_n x = 0$ has two roots which pass from reality, through equality, to unreality, *i.e.*, if $F_n r$ is positive, all the roots of $F_n x = 0$ are unreal.

The conditions therefore necessary to be fulfilled in order that $F_n x = 0$ may be incapable of having more than one real root may be thus stated.

1. The coefficients a_0, a_2, a_4, &c. must all be positive.

2. C_2, C_4, C_6, &c. must all be positive.

3. Or if the real root in $F_1 x = 0$, $F_3 x = 0$, &c. be respectively r_1, r_3, &c., then $F_2 r_1$, $F_4 r_3$, &c. must all be positive.

The second and third of these sets of conditions must of course involve the same principle, so that if one holds good the other must hold good also. We can however always obtain the second set of conditions by mere elimination, but the third set, except in particular cases, will require the solution of equations.

Professor SYLVESTER's Question may now be solved by proving that if, for any positive integral value of n, $\phi x = 0$ is incapable of having more than one real root, this must also hold good for the next value of n, and therefore generally, provided ν is positive or $< -n$.

First, let n be odd, and let $\phi' x = \dfrac{d\phi x}{dx}$; then, by hypothesis, $\phi' x = 0$ can have no real root, and therefore $\phi x = 0$ can only have one real root.

Next, let n be even, then, by hypothesis, $\phi' x = 0$ has only one real root. Let that root be α; then we have

$$0 = 1 + (\nu+1)\alpha + \frac{(\nu+1)(\nu+2)}{1\,.\,2}\alpha^2 + \,.\,.\, + \frac{(\nu+1)\,.\,.\,(\nu+n-1)}{1\,.\,2\,.\,.\,(n-1)}\alpha^{n-1}, \text{ and}$$

$$\phi\alpha = 1 + \nu\alpha + \frac{\nu(\nu+1)}{1\,.\,2}\alpha^2 + \,.\,.\, + \frac{\nu(\nu+1)\,.\,.\,(\nu+n-1)}{1\,.\,2\,.\,.\,n}\alpha^n,$$

therefore, by subtracting, we obtain

$$\phi\alpha = -\left(\alpha + (\nu+1)\alpha^2 + \frac{(\nu+1)(\nu+2)}{1\,.\,2}\alpha^3 + \,.\,.\, + \frac{(\nu+1)\,.\,.\,(\nu+n-2)}{1\,.\,2\,.\,.\,(n-2)}\alpha^{n-1}\right) + \frac{\nu(\nu+1)\,.\,.\,(\nu+n-1)}{1\,.\,2\,.\,.\,n}\alpha^n$$

$$= \left\{\frac{(\nu+1)\,.\,.\,(\nu+n-1)}{1\,.\,2\,.\,.\,.\,.\,(n-1)} + \frac{\nu(\nu+1)\,.\,.\,(\nu+n-1)}{1\,.\,2\,.\,.\,n}\right\}\alpha^n = \frac{(\nu+1)(\nu+2)\,.\,.\,(\nu+n)}{1\,.\,2\,.\,.\,n}\alpha^n$$

which, since n is even, is necessarily positive, provided ν be positive or $< -n$.

But it has been shown that if $\phi' x = 0$ has but one real root, and if that root, substituted in ϕx, gives a positive result, the roots of $\phi x = 0$ must be all unreal; hence when n is even, $\phi x = 0$ must have all its roots unreal. If therefore the hypothesis holds good for $n = 2$, which is easily shown to be the case, it must hold good for $n = 3$ and $n = 4$, &c., and therefore for $n = 5$ and $n = 6$, and so on generally.

Again, let ν be indefinitely great, then ϕx is reduced to

$$1 + \nu x + \frac{\nu^2 x^2}{1\,.\,2} + \cdots + \frac{\nu^n x^n}{1\,.\,2\,.\,.\,n},$$

and if x be assumed indefinitely small, νx may be assumed to equal any finite quantity. For νx put x, and for ϕx put fx; then $fx = 0$ must be incapable of having more than one real root.

1817. (Proposed by M. COLLINS, B.A.)—In lines of the third order, prove that the locus of middle points of chords parallel to an asymptote which does not *cut the curve*, is a straight line; but when the asymptote cuts the curve, show that the locus then becomes a hyperbola.

Solution by PROFESSOR CREMONA.

Ce théorème n'est qu'une corollaire très-simple d'une propriété connue (*Introd. ad una Teoria geom. delle curve piane*, 139). Les cordes d'une cubique plane parallèles à une asymptote ont leurs milieux dans une conique (la conique polaire du point à l'infini sur l'asymptote), qui a cette asymptote en commun avec la cubique, et est par conséquent une hyperbole. Cela arrive toujours si le point à l'infini sur l'asymptote est un point ordinaire de la cubique. Mais si ce point est un point d'inflexion (ce qui revient à supposer que l'asymptote ne rencontre pas la cubique à distance finie), la conique polaire se décompose en deux droites, dont l'une est l'asymptote même, et l'autre, lieu effectif des points-milieux des cordes, est nommée *polaire harmonique* du point d'inflexion.

1835. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.)—Three lines being drawn at random on a plane, determine the probability that they will form an acute triangle.

Solution by the PROPOSER.

The condition in the question, abstractedly considered, depends on the circumstances of direction only without any regard to position or magnitude. And every possible direction will be comprised in an angular range of two right angles, since lines drawn in opposite directions may here be properly treated as identical, the idea being, in fact, that of linear direction apart from any consideration of an origin or zero point of linear magnitude. For simplicity in estimating the angles, let a right angle be taken as the unit of measurement; and suppose two lines, drawn at random, and prolonged indefinitely, to intersect each other at supplemental angles x, $2-x$, where x may be any positive value less than unity. Then in order that the triangle completed by drawing a third line may be acute, the vertical angle must evidently be the acute angle x; and the sum of the second and third angles of the triangle must therefore be equal to $2-x$. Moreover the second and third angles must also be both of them acute; and therefore it follows that each of them must exceed $1-x$ and not exceed 1. The direction of the third line is thus restricted to an angular range of x, within which the three angles of the resulting triangle are severally acute. Therefore, as the entire unlimited range in the direction of this line is 2, and as the elements comprised in those ranges are obviously equally admissible, the probability of an acute triangle, under the hypothesis of the assumed lines, is equal to $\frac{1}{2}x$. Hence, as the two lines originally assumed may indifferently have every value of x from 0 to 1, the required absolute probability of an acute triangle is obtained by multiplying by dx and integrating between those limits, and is $\frac{1}{4}$.

NOTE.—The probability involved in this question is the same as that in which the three angular points are taken at random in the circumference of a given circle; as is evident from the consideration that, when an angular point is supposed to pass over equal arcs of the circle, the directions of the adjacent sides will describe equal angles.

II. *Solution by* J. M. WILSON, M.A.; E. MCCORMICK; M. COLLINS, B.A.; E. FITZGERALD; *and others.*

The probability of the acute angle made by two lines lying between θ and $\theta + d\theta$ is $\frac{2d\theta}{\pi}$, and all lines whose directions lie between the perpendiculars to the two lines will make acute triangles with them; hence in this case the probability will be $\frac{\theta}{\pi}$. Therefore the probability required will be

$$p = \int_0^{\frac{1}{2}\pi} \frac{\theta}{\pi} \cdot \frac{2d\theta}{\pi} = \frac{1}{4}.$$

1873. (Proposed by C. W. MERRIFIELD, F.R.S.)—Assuming that all lives are of equal duration, what must that duration be, in order that the births, deaths, and consequent increase or decrease of population, may remain unchanged?

Solution by the PROPOSER; *and* J. H. TAYLOR, B.A.

Let the ratio of births to a unit of population be b, and that of the deaths d, then the rate of increase is $r = b - d$; and let t be the length of the period over which the b and d are reckoned.

If b were the same as d, or $r = 0$, we should have simply $l = \frac{t}{d}$, l being the duration of life.

If we consider the increase in the period t to be r, then the momentary rate of increase will be $\log_\epsilon (1+r)$; for, if we consider the instantaneous rate of increase to be ρ, we have $\epsilon^\rho = 1 + r$. We must therefore affect d with the factor $\frac{\log(1+r)}{r}$ to bring it to the momentary standard; for the equation $r = b - d$ is true for any period, and at the limit also.

Hence we have $l = \dfrac{rt}{d \log_\epsilon (1+r)}$.

1877. (Proposed by J. GRIFFITHS, M.A.)—Let P be the intersection of the three perpendiculars; O the centre of the circumscribed circle (radius

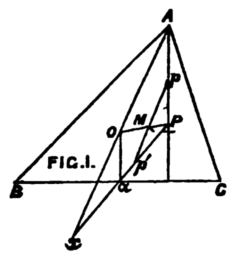
$= R$); α, β, γ the middle points of the sides, of any triangle ABC. On the segments PA, PB, PC let the three points p, q, r be taken, such that $Pp = \frac{1}{n} . PA$, $Pq = \frac{1}{n} . PB$, $PR = \frac{1}{n} . PC$; and on $P\alpha$, $P\beta$, $P\gamma$ three other points p', q', r', such that $Pp' = \frac{2}{n} . P\alpha$, $Pq' = \frac{2}{n} . P\beta$, $Pr' = \frac{2}{n} . P\gamma$. Prove (1) that the lines pp', qq', rr' intersect on the line PO in a point M, such that $PM = \frac{1}{n} . PO$; (2) that the six points in question lie on a circle whose centre coincides with M, and whose radius $= \frac{1}{n} . R$; (3) that this circle will touch the circle inscribed in the triangle, if $\frac{1}{n} = \frac{1}{2}$ or $= 1 + \frac{r^2}{\rho^2}$, where r, ρ are the radii of the inscribed and self-conjugate circles of the triangle.

Solution by the PROPOSER; J. H. TAYLOR, B.A.; *and others.*

1. Produce AO, $P\alpha$ (Fig. 1) to meet in x; and join $O\alpha$; then, by similar triangles, we have

$$xO : xA = \alpha O : PA = 1 : 2,$$

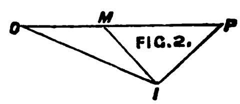
therefore O is the point of bisection of Ax, and α that of Px. Again, since $Pp : PA = Pp' : Px$, the line pp' is parallel to Ax, and must therefore cut PO in a point M such that $PM = \frac{1}{n} . PO$ and $pM = p'M = \frac{1}{n} . AO$. In the same way qq' and rr' may be shown to intersect the line PO in M.



2. Hence, since $AO = BO = CO = R$, the six points p, p'; q, q'; r, r' will lie on a circle whose centre is M and radius $\frac{1}{n}$ R.

3. Let I (Fig. 2) be the centre of the inscribed circle of the given triangle; then, if the circles (M) and (I) touch each other, we must have

$$MI = \frac{1}{n} . R \pm r.$$



$$\text{But } \frac{PO^2 + PI^2 - OI^2}{2PO . PI} = \cos OPI = \frac{PM^2 + PI^2 - MI^2}{2PM . PI}, \text{ and } PM = \frac{1}{n} . PO;$$

therefore $PO^2 + PI^2 - OI^2 = n\,(PM^2 + PI^2 - MI^2)$; also $OI^2 = R^2 - 2Rr$,

moreover $\quad PI^2 = 2r^2 - 4R^2 \cos A \cos B \cos C = 2r^2 + \rho^2$,

and $\quad PO^2 = R^2 - 8R^2 \cos A \cos B \cos C = R^2 + 2\rho^2$;

$$\text{therefore} \quad (2r^2 + 3\rho^2 + 2Rr) = n\left\{\frac{R^2 + 2\rho^2}{n^2} + 2r^2 + \rho^2 - \left(\frac{R}{n} \pm r\right)^2\right\},$$

$$\text{or} \quad (2r^2 + 3\rho^2 + 2Rr)\,n = (r^2 + \rho^2)\,n^2 \mp 2Rrn + 2\rho^2.$$

If we take the + sign, we shall have $(r^2+\rho^2)\,n^2+2\rho^2=(2r^2+3\rho^2)\,n$,

whence we obtain $$\frac{1}{n}=1+\frac{r^2}{\rho^2},\ \text{or}\ \frac{1}{2}.$$

The other values of n for which the circles touch each other are evidently given by the equation $(r^2+\rho^2)\,n^2+2\rho^2=(2r^2+3\rho^2+4Rr)\,n$.

When $n=\frac{1}{2}$, the circle becomes the well-known nine-point circle.

1879. (Proposed by T. COTTERILL, M.A.)—If forces represented by the sides of a plane hexagon taken in order are in equilibrium, the directions of the sides of the two triangles formed by joining alternate points of the hexagon are in involution.

Solution by the REV. R. TOWNSEND, M.A.

Denoting the three pairs of opposite vertices of the hexagon by A and A′, B and B′, C and C′; and replacing the three pairs of forces AB′ and AC′, BC′ and BA′, CA′ and CB′, acting at the three vertices A, B, C of either triangle, by their three resultants AX, BY, CZ, which are evidently parallel to the three corresponding sides B′C′, C′A′, A′B′ of the other; then since the three lines AX, BY, CZ, as representing by hypothesis three forces in equilibrium, are concurrent, therefore the theorem follows; every three concurrent lines through the three vertices of any triangle determining with the three parallels through the point of concurrence to the three sides of the triangle a system of six rays in involution. (Townsend's *Modern Geometry*, Art. 368, Ex. 2.)

Since, when a number of forces in a plane are represented in magnitude and direction by the several sides of a polygon in the plane, the sum of their moments round any point in the plane is represented by double the area of the polygon (*Modern Geometry*, Art. 118); it follows at once, as an immediate corollary from the above, that *when the area of a plane hexagon* = 0, *the directions of the sides of the two triangles determined by its two triads of alternate vertices are in involution.*

1909. (Proposed by the Rev. R. H. WRIGHT, M.A.)—If λ and λ' be the angles which any two conjugate diameters AB and CD of an ellipse subtend at any point P in the curve, and α the angle which either axis subtends at an extremity of the other axis; prove that $\cot^2\lambda+\cot^2\lambda'=\cot^2\alpha$.

Solution by S. W. BROMFIELD; J. H. TAYLOR, B.A.; *the* PROPOSER; *and others.*

Let the points P, A, C be respectively denoted by $(a\cos\theta,\ b\sin\theta)$, $(a\cos\phi,\ b\sin\phi)$, $(a\cos\phi',\ b\sin\phi')$, where $\phi'=90^\circ+\phi$; then the tangents

of the angles which AP and BP make with the major axis are $\frac{b}{a} \cdot \frac{\sin\phi \mp \sin\theta}{\cos\phi \mp \cos\theta}$;

$$\text{therefore } \tan\lambda = \frac{2ab}{a^2-b^2} \cdot \frac{\sin(\theta-\phi)}{\cos^2\phi - \cos^2\theta} = \frac{2ab}{a^2-b^2} \cdot \frac{1}{\sin(\theta+\phi)};$$

$$\text{similarly } \tan\lambda' = \frac{2ab}{a^2-b^2} \cdot \frac{1}{\sin(\theta+\phi')} = \frac{2ab}{a^2-b^2} \cdot \frac{1}{\cos(\theta+\phi)},$$

$$\text{therefore} \qquad \cot^2\lambda + \cot^2\lambda' = \left(\frac{a^2-b^2}{2ab}\right)^2 = \cot^2\alpha.$$

Angular and Linear Notation.

A Common Basis for the *Bilinear* (a transformation of the *Cartesian*), the *Trilinear*, the *Quadrilinear*, &c., Systems of Geometry. By H. McColl.

Def. 1. Let any two straight lines X and Y intersect in a point S. From S as centre and with any radius describe a circle cutting X in the point A, and Y in the point B. The angle ASB is proportional to the arc BA, measured in the *positive* (or *unscrewing*) direction of the circumference *from the point* B *to the point* A; and the angle BSA is proportional to the remaining portion of the circumference, measured in the *same* (positive) direction *from the point* A *to the point* B. Hence, if θ be the circular measure of ASB, $2\pi-\theta$ will be the circular measure of BSA.

Def. 2. The *Point of Reference* (denoted by P) is a point taken in a plane indicating the directions in which angles and distances are to be measured. It must be understood *that no line in its plane, mentioned in the course of an investigation, passes through it, and that it is situated on the positive side of every line in the investigation.* In other respects its position is arbitrary.

Def. 3. Let X, Y, Z, U, V, &c., be any straight lines in the same plane. Let p be the general expression for every point in some locus W. Then x, y, z, u, v, &c., denote respectively *the lengths of the perpendiculars from* p *upon the straight lines* X, Y, Z, U, V, &c. Any one of these distances, say x, is understood to be *positive* when p and P are *on the same side* of the corresponding line X, and *negative* when *on opposite sides of* X. Hence, the values of x, y, z, u, v, &c., vary with the position of p. The perpendiculars x, y, z, &c., are called the *coordinates* of the point p.

Def. 4. The symbols $\mathbf{x}\,\mathbf{y}$ and $\mathbf{x}{\cdot}\mathbf{y}$, are called respectively the first and second inclinations of X to Y. Interpretation:—Let X and Y intersect in S. From S as centre, and with any radius, describe a circle cutting X *on the positive side* of Y in the point $\mathbf{x}_1$ and *on the negative side* of Y in the point $\mathbf{x}_2$, and also cutting Y *on the positive side* of X in the point y. Then $\mathbf{x}\,\mathbf{y}$ denotes the angle $\mathbf{x}_1\,\mathbf{s}\,\mathbf{y}$, and $\mathbf{x}{\cdot}\mathbf{y}$ denotes the angle $\mathbf{x}_2\,\mathbf{s}\,\mathbf{y}$; these angles being interpreted as in Def. 1. Hence $\mathbf{y}\,\mathbf{x} = 2\pi - \mathbf{x}\,\mathbf{y}$ and $\mathbf{x}{\cdot}\mathbf{y} = \mathbf{x}\,\mathbf{y} \pm \pi$, the positive or negative sign to be taken according as $\mathbf{x}\,\mathbf{y}$ is less or greater than π. It will be observed that the sine, the cosine, the tangent, &c., of $\mathbf{x}{\cdot}\mathbf{y}$ are not affected by the double sign.

Def. 5. When every point in a locus W is subject to any condition $f(x, y, z, u, v, \&c.) = 0$, this is denoted by $(w) = f(x, y, z, u, v, \&c.)$, which must be considered as a brief expression for the sentence,—"The equation to the locus W is $f(x, y, z, u, v, \&c.) = 0$." The symbol (w) may be called the *zero expression of the line* (or locus) W. When the zero expression contains *two* variable coordinates, it is called *bilinear;* when it contains *three*, *trilinear*, and so on. Similarly, an investigation *considered as a whole* may be called *bilinear, trilinear, quadrilinear*, &c., according to the number of variable coordinates employed throughout it. Every *zero expression*, for instance, may be *bilinear*, while the whole investigation is *quintilinear.*

The following are a few applications of this notation.

1. Let X, Y, Z be any three straight lines in the same plane. Whatever be the position of the point of reference P, any one of the three, say Z, is connected with the other two by the relation

$$(z) = x \sin \mathrm{y\,z} + y \sin \mathrm{z\,x} - z' \sin \mathrm{x\,y},$$

in which z' denotes the perpendicular upon Z from the intersection of X and Y.

When either of the coefficients sin y z, sin z x, becomes zero, the other becomes numerically equal to sin x y; so that we shall have

$$(z) = y \pm z'; \quad (z) = x \pm z';$$

the signs depending upon the position of P. The first is the equation to Z when parallel to Y; the second is the equation to Z when parallel to X. When $z' = 0$, we shall have

$$(z) = x \sin \mathrm{y\,z} + y \sin \mathrm{z\,x},$$

which is the equation to Z when passing through the intersection of X and Y. When, in the last equation, the coordinates x and y are of the same sign *for the same position of p*, we shall see that sin y z and sin z x have opposite signs; and when x and y have opposite signs, sin y z and sin z x have the same sign. When sin y z and sin z x become *numerically* equal, the last equation becomes

$$(z) = x \pm y.$$

An examination of Defs. 3 and 4 will make it clear that when the *positive* sign is taken Z bisects the angle x'y, and that when the *negative* sign is taken Z bisects x y.

By means of this equation we can immediately show that, *whatever be the position of* P, the bisectors of the three angles x y, y z, z x *meet in a point.* For if U, V, W be respectively the three bisectors, we shall have

$$(u) = x - y; \quad (v) = y - z; \quad (w) = z - x;$$

and the values of x and y which satisfy (u) and (v) simultaneously will also satisfy (w). The point at which the three bisectors meet will be found within or without the triangle intercepted by the lines X, Y, Z, according as the point of reference P is within or without this triangle. The solution of this problem, therefore, on the principle of *Angular and Linear Notation*, is more comprehensive than that usually given.

2. Let X, Y, Z be the sides of any triangle, understood to have the same signs respectively as their proportionals sin y z, sin z x, sin x y; and let

z' be the perpendicular upon Z from the intersection x y. Whatever be the position of P we shall have

$$(z) = Xx + Yy - Zz',$$

in which it will be observed that $Zz' =$ twice area of triangle. If P be taken within the triangle, and U, V, W be respectively the bisectors of Z, X, Y from the opposite angles, we shall have

$$(u) = Xx - Yy\,; \quad (v) = Yy - Zz\,; \quad (w) = Zz - Xx,$$

which shows that U, V, and W meet in a point.

Let X, Y, Z, U be any straight lines in the same plane, we shall have, as in the usual trilinear system,

$$(u) = lx + my + nz,$$

in which l, m, n are constant quantities.

In Cartesian language, let x, y be the coordinates of any point p referred to oblique axes. For "axis of y" read "line X;" for "axis of x" read "line Y;" interpret x and y as in Def. 3, that is, as the perpendiculars from p upon X and Y respectively: then, if a and b be the values of x and y in the Cartesian language, and a_1 and b_1 be the values of x and y when interpreted as in Def. 3, we shall have $a_1 = a \sin \mathrm{x\,y}$, and $b_1 = b \sin \mathrm{x\,y}$.

Hence, any equation expressed in Cartesian language may immediately be transformed into another expressed in bilinear (perpendicular) coordinates. For example, the equation to the circle referred to oblique axes is

$$(x-a)^2 + (y-b)^2 + 2\,(x-a)\,(y-b)\cos\omega - c^2 = 0,$$

in which a, b are the coordinates of the centre, c denotes the radius, and ω the inclination of the axes. The same equation expressed in bilinear (perpendicular) coordinates is

$$(z) = (x-a_1)^2 + (y-b_1)^2 + 2\,(x-a_1)\,(y-b_1)\cos \mathrm{x\,y} - c_1^2,$$

in which Z denotes the circumference, x and y are the perpendiculars from *any* point in Z upon the lines X and Y respectively, a_1 and b_1 are the perpendiculars from the centre upon X and Y respectively, and c_1 is the radius multiplied by $\sin \mathrm{x\,y}$.

The advantages of the Notation here proposed may be summed up as follows:—

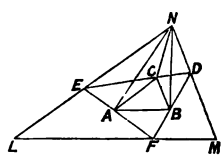
By substituting the more comprehensive notion of a *Point of Reference* for that of a fixed *Origin*, adopting *perpendicular* coordinates (denoted by Italic letters) in all cases, and removing all the unnecessary restrictions imposed by the conventions of fixed *axes*, *triangles*, and *quadrilaterals* of reference, the *Linear* Notation will enable us to unite the *Cartesian* and modern *Trilinear*, &c. methods in one harmonious system of Analytical Geometry.

The *Angular* Notation, on the other hand, supplies us with brief and *distinct* expressions for *all* the various angles resulting from the intersections of any number of straight lines in the same plane. The symbols x y. y x. x·y and y·x are four distinct expressions for the four different ways in which the angles resulting from the intersection of X and Y may be considered: and the point of reference P is an infallible index as to the particular angle denoted by any one of the symbols. An instance of the *comprehensiveness* of the Notation has already been given in the statement that, *whatever be the position of* P, the bisectors of x y, y z, z x meet in a point. And the fact that, on this principle, no two straight lines which are not parallel can make equal angles with the same straight line, is an instance of the *precision* of the Notation.

The Angular and Linear Notation may also be applied to Solid Geometry. P will in this case denote the positive side of every plane under investigation, and the only condition with regard to its position is that none of the planes spoken of shall pass through it.

1263. (Proposed by N'IMPORTE.)—Dans un triangle quelconque ABC, on mène, à partir des sommets, trois droites dans des directions quelconques; ces trois droites par leurs intersections donnent naissance à un second triangle DEF. Par les sommets de celui-ci on fait passer des droites respectivement parallèles aux côtés du triangle ABC; on obtient ainsi un troisième triangle LMN. Démontrer que la surface du triangle DEF est moyenne proportionnelle entre les surfaces des deux autres.

Solutions (1) *by* W. HOPPS; J. McDOWELL, B.A.; *and* S. WATSON; (2) *by* S. BILLS; *and others.*



1. Let $\Delta_1, \Delta_2, \Delta_3$ denote the respective areas of the triangles ABC, DEF, LMN; and P_1, P_3 the perpendiculars from the vertices C, N of the triangles Δ_1, Δ_3 on the opposite sides AB, LM respectively. Join NA, NB, NC. Then by parallels we have Δ NAC = EAC, and Δ NBC = DBC; therefore $\Delta_2 =$ NAFB.

Hence $\Delta_1 : \Delta_2 = P_1 : P_3$; also $\Delta_3 : \Delta_1 = P_3^2 : P_1^2$;

therefore $\Delta_3 : \Delta_2 = P_3 : P_1 = \Delta_2 : \Delta_1$.

2. *Otherwise:* take A for origin of *Cartesian coordinates*, and AB, AC for the axes of x and y; and put $AB = r$, $AC = s$.

Let the equations of EF, FD, DE be respectively

$$x = my, \quad x - r = ny, \quad x = p(y - s).$$

Then the respective coordinates of D, E, F will be

$$\left(\frac{p(ns+r)}{p-n}, \frac{ps+r}{p-n}\right), \left(\frac{mps}{p-m}, \frac{ps}{p-m}\right), \left(\frac{mr}{m-n}, \frac{r}{m-n}\right).$$

Hence the equations of LM, MN, NL are found to be

$$y = \frac{r}{m-n}, \quad x = \frac{mps}{p-m}, \quad \frac{x}{r} + \frac{y}{s} = \frac{p(ns+r)}{r(p-n)} + \frac{ps+r}{s(p-n)}.$$

Now $\Delta_1 = \frac{1}{2} rs \sin A$; also from the above we readily find

$$\Delta_2 = \frac{\left\{(p-m)r - ps(m-n)\right\}^2 \sin A}{2(m-n)(n-p)(p-m)}, \quad \Delta_3 = \frac{\left\{(p-m)r - ps(m-n)\right\}^4 \sin A}{2rs(m-n)^2(n-p)^2(p-m)^2};$$

therefore $\Delta_1 \Delta_3 = \Delta_2^2$, or $\Delta_1 : \Delta_2 = \Delta_2 : \Delta_3$.

[As an exercise for our junior readers, we add a proof by *Trilinear Coordinates*:—

Let $\alpha = 0$, $\beta = 0$, $\gamma = 0$ be the equations of the sides of the triangle DEF, and a, b, c the lengths of these sides respectively. Then the triangle ABC has one of its vertices on each of the sides of the triangle of reference DEF (or on the prolongations of these sides); hence it may be readily shown (see *Educational Times* for January and April, 1854, Quest. 663) that the equations of the sides of the triangle ABC may be written

$$\alpha + n\beta + \frac{\gamma}{m} = 0, \quad \frac{\alpha}{n} + \beta + l\gamma = 0, \quad m\alpha + \frac{\beta}{l} + \gamma = 0.$$

The equations of the sides of the triangle LMN, drawn through the vertices of the triangle of reference parallel to the sides of the triangle ABC, are

$$(na-b)\,\beta + \left(\frac{a}{m} - c\right)\gamma = 0, \quad (lb-c)\,\gamma + \left(\frac{b}{n} - a\right)\alpha = 0,$$

$$(mc-a)\,\alpha + \left(\frac{c}{l} - b\right)\beta = 0.$$

From the foregoing equations of the sides of the triangles ABC, LMN, we find, by applying the formula proved in the Solutions of Quest. 1733 (*Reprint*, Vol. IV., pp. 53—56), that the areas of these triangles are

$$\Delta_1 = \frac{abc\,(lmn-1)\,\Delta_2}{(lb-c)\,(mc-a)\,(na-b)}, \quad \Delta_3 = \frac{(lb-c)\,(mc-a)\,(na-b)\,\Delta_2}{abc\,(lmn-1)};$$

where Δ_2 is the area of the triangle of reference DEF.

Hence we have $\quad \Delta_1\Delta_3 = \Delta_2^2$, or $\Delta_1 : \Delta_2 = \Delta_2 : \Delta_3$].

1720. (Proposed by M. W. Crofton, B.A.)—A given curve moves in its own plane, without rotation, so as always to pass through a fixed point A; in any of its positions draw a tangent at A, and a second tangent cutting this at right angles; and find the envelope of the latter.

Solution by the Proposer.

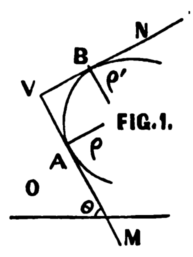


FIG. 1.

Let VA, VB (Fig. 1) be two perpendicular tangents to a given curve; let $VA = T$, $VB = T'$; let ρ, ρ' be the radii of curvature at A, B; let θ = inclination of the tangent VA to a fixed axis, and let the curve be represented by the *intrinsic* equation $\rho = f(\theta)$. We shall have (see Quest. 1622, *Reprint*, Vol. III., p. 75)

$$\frac{d^2T}{d\theta^2} + T = \rho' - \frac{d\rho}{d\theta}.$$

Suppose now the right angle MVN to revolve, always touching the curve in two points; and in

each position let the curve be transferred without rotation so that the point of contact A shall fall upon a fixed point O (Fig. 2). The tangent VN will envelope some curve, and T, or OV, will be the perpendicular from the fixed point O on the tangent to the envelope. Now if R = radius of curvature of the envelope at P, we have, (since in any curve $\rho = \dfrac{d^2p}{d\theta^2} + p$)

$$R = \frac{d^2T}{d\theta^2} + T = \rho' - \frac{d\rho}{d\theta}, \therefore R = f\left(\theta + \frac{\pi}{2}\right) - \frac{d}{d\theta} f(\theta).$$

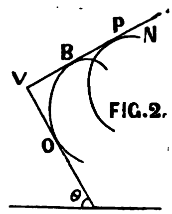


FIG. 2.

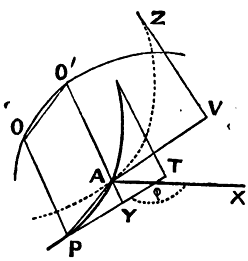
If ϕ = inclination of tangent of envelope to the axis, its intrinsic equation is therefore

$$R = f(\phi) - \frac{d}{d\phi} f\left(\phi - \frac{\pi}{2}\right).$$

This may be applied to various curves. If the moving curve is a circle, the envelope is an equal circle. If the moving curve is a cycloid, its equation will be $\rho = -4a \cos\theta$; and we shall find the equation of the envelope to be R=0, that is, a point; so that in this case the perpendicular tangent passes through a fixed point: a result which may be verified by other methods.

II. *Solution by* F. D. Thomson, M.A.

1. Let AP, AZ be two positions of the moving curve which always passes through A and moves without rotation; then if P be the point which has moved up to A, O any other point which has moved to O′, AP is equal and parallel to OO′, and therefore AO′ is equal and parallel to PO. Hence any point O describes a curve about the fixed point A similar and equal to the original curve, but turned in an opposite direction; also the point O corresponds to A, and O′ to P on the original curve, and the tangent at O′ will be parallel to the tangent at P. Hence, in order to construct the tangent at A to the curve AZ, construct the locus of the point O, let O′ be the new position of O, and draw a line AV parallel to the tangent at O′.



2. Next to find the envelope of the tangent VZ perpendicular to AV. Draw AY perpendicular to PY. Then from the figure we see that AV = PY + perpendicular on tangent to AP which is perpendicular to PY. Hence if $p = f(\phi)$ be the equation to AP, we have AV = PY $+ f(\phi - \frac{1}{2}\pi) = f'(\phi) + f(\phi - \frac{1}{2}\pi)$, since PY $= \dfrac{dp}{d\phi} = f'(\phi)$. Thus putting AV $= p'$, $\phi - \frac{1}{2}\pi = \phi'$, we have $p' = f'(\phi' + \frac{1}{2}\pi) + f(\phi')$; hence, dropping the accents, the tangential polar equation of the envelope is $p = f(\phi) + f'(\phi + \frac{1}{2}\pi)$.

Example. Let the curve be the equiangular spiral $r = ae^{\theta \cot \alpha}$, the fixed point A being the pole. Then $\phi = \theta + \alpha$, and $p = r \sin \alpha = a \sin \alpha\ e^{(\phi -) \alpha \cot \alpha}$; therefore $f'(\phi) = a \cos ae^{(\phi - \alpha) \cot \alpha}$; hence the equation of the envelope is

$$p = a\,(\cos \alpha\ e^{\frac{1}{2}\pi \cot \alpha} + \sin \alpha)\ e^{(\phi - \alpha) \cot \alpha},$$

which is that of an equiangular spiral.

1783. (Proposed by R. TUCKER, M.A.)—Eliminate θ between each of the following sets of equations:

$$\text{(I)} \ldots \left\{ \begin{array}{l} X \cos(\theta - A) + Y \cos \theta = 2R \sin(\theta + C) \cos \theta \cos(\theta - A) \ldots . (1) \\ X \sin(\theta - A) + Y \sin \theta = -2R \cos(\theta + C) \sin \theta \sin(\theta - A) \ldots (2) \end{array} \right\};$$

$$\text{(II)} \ldots \left\{ \begin{array}{l} Y - X \cot(B + \tfrac{1}{2}\theta) = R \left\{ \cos C - \sin(C - \theta) \cot(B + \tfrac{1}{2}\theta) \right\} \ldots (1) \\ Y + X \tan(B + \tfrac{1}{2}\theta) = R \left\{ \cos C - \sin(C - \theta) \tan(B + \tfrac{1}{2}\theta) \right\} \ldots (2) \end{array} \right\};$$

where A, B, C are the angles of a triangle, and R the radius of its circumscribing circle.

Solution by the PROPOSER; S. BILLS; *and others.*

I. Here, from $\left\{(1) . X \sin \theta + (2) . X \cos \theta\right\}$ we obtain

$$X \sin A \cot 2\theta = x \cos A + Y - R \sin B \ldots \ldots \ldots \ldots (\alpha).$$

Again, from $\left\{(1) . X \sin(\theta - A) + (2) . X \cos(\theta - A)\right\}$ we have

$$\cot 2\theta\,(X \sin 2A + Y \sin A - R \sin C \sin 2A) = X \cos 2A + Y \cos A - R \sin C \cos 2A \ldots (\beta).$$

Eliminating θ between (α) and (β), we get

$$X^2 + 2XY \cos A + Y^2 - \tfrac{1}{2}(c + 2b \cos A)\,X - \tfrac{1}{2}(b + 2c \cos A)\,Y + \tfrac{1}{2}bc \cos A = 0.$$

II. Eliminating X and Y successively, we get

$$Y - R \cos C = -R \sin(C - \theta) \sin(2B + \theta),\quad X = R \sin(C - \theta) \cos(2B + \theta);$$

$$\therefore X^2 + (Y - R \cos C)^2 = R^2 \sin^2(C - \theta),\ \tan(2B + \theta) = X^{-1}(R \cos C - Y) \ldots (\gamma, \delta).$$

From this latter equation (δ), we have

$$\sin \theta = \frac{(Y - R \cos C) \cos 2B + X \sin 2B}{\left\{X^2 + (Y - R \cos C)^2\right\}^{\frac{1}{2}}},$$

$$\cos \theta = \frac{(Y - R \cos C) \sin 2B - X \cos 2B}{\left\{X^2 + (Y - R \cos C)^2\right\}^{\frac{1}{2}}}.$$

Substituting in (γ) there results

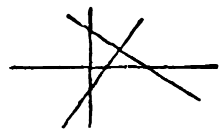
$$X^2 + (Y - R \cos C)^2 = R\left\{(Y - R \cos C) \cos(A - B) - X \sin(A - B)\right\}.$$

[We may remark that the given equations represent the two feet-perpendicular lines in Quest. 1649 (*Reprint*, Vol. III., p. 58). In (I) the axes are AB, AC; in (II) they are lines through O parallel and perpendicular to AB. The result in each case is the nine-point circle of the given triangle, thus affording another proof of the theorem.]

On the Four-point and similar Geometrical Chance Problems.

By J. M. Wilson, M.A., F.G.S.

The four-line problem proposed by me as Quest. 1868 was to find the chance that if three lines were drawn at random in an infinite plane, a fourth line drawn at random would intersect the triangle formed by the other three.



If four lines are drawn, it will be found that two of them always intersect the triangle formed by the other three, and since the lines are equally at random the chance required is $\frac{1}{2}$. To put the solution in another form; if a blind-folded man were told that three lines were drawn on a large piece of paper, and was asked to draw a fourth, and to decide the probability of its intersecting the triangle; he might draw the line, and on examining the figure would see that if *his* line were one of two, it would intersect the triangle, and if not, it would not; and having nothing to guide him as to which was *his* line, would decide the probability to be $\frac{1}{2}$.

The four-point problem is to find the chance of four points at random in an infinite plane forming a reentrant quadrilateral. I argue thus:—Four lines at random determine three sets of four random points, which form one reentrant and two convex quadrilaterals, and therefore the chance is $\frac{1}{3}$.

This result follows from, and is as certain as, the axiom, that a random point may be looked on as the intersection of two random lines. And the reasoning may be justified as before. A blind-folded man is told there are four points on a sheet of paper, and is asked the chance of their being the angles of a reentrant quadrilateral. He requests that they may be joined by lines, so that each point has two and only two lines passing through it. The lines are produced indefinitely, and the only condition that depends on the space being supposed infinite is that the lines do intersect on the sheet of paper before him. He now examines the figure, and it is impossible for him to tell from which of the three quadrilaterals before him the four lines originated. And since only one, and always one of the three is reentrant, he must reason that the chance is $\frac{1}{3}$. (See the above Figure.)

I see here no uncertainty introduced by the space being infinite, and there is no comparison of infinities which is fallacious; it is equally true for this sheet of paper, if it is granted that the figure is such as can be drawn upon it.

Below, however, are some remarks made on this Solution by Dr. Ingleby, with whose judgment I have the misfortune to differ. He seems to me to have proved that the probability is less than $\frac{1}{3}$, by a method which is incapable of assigning how much less. I can attach no meaning whatever to the last line, on which his whole argument depends.

On a Problem in the Theory of Chances.

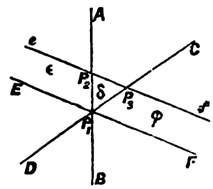
By C. M. Ingleby, LL.D.

Four points being placed at random in a plane, what is the chance of one of them falling within the triangle formed by the other three?

This problem has been variously solved by Professors Cayley, Sylvester, and Price; and above is a fourth solution by Mr. J. M. Wilson of Rugby, the Senior Wrangler of 1859. Strange to say, no two of these four results are alike. The value assigned by Mr. Wilson to the required chance is $\frac{1}{3}$.

Now I submit that, whatever be the true value, this at least is amenable to a very simple *reductio ad absurdum.*

If two points be taken at random, and three straight lines, lying in a plane passing through both, mutually intersect in one of them, the chance that the other will be found in one of the alternate angles α, β, γ is $\dfrac{\alpha+\beta+\gamma}{2(\alpha+\beta+\gamma)}=\frac{1}{2}$. This, at least, will not be disputed. We are indeed comparing two infinities, but two infinities of which we know all we need, viz., their ratio to each other. This result will be of use in dealing with the problem before us.



Take any two of the four points (P_1, P_2) and draw a straight line AB through both. Then the chance that P_3 (either of the other two points) lies out of AB may be taken as unity. Let it be to the right of AB. Through P_1, P_3 draw the straight line CD, and through P_2, P_3 draw the straight line *ef*. Also through P_1 draw the straight line EF parallel to *ef*. Let the area $EP_1P_2e=\epsilon$, $FP_1P_3f=\phi$, and $P_1P_2P_3=\delta$. Now it is plain that in order that one of the four points may fall within the triangle formed by the other three, P_4 must fall within one of the areas $P_1P_2P_3$, AP_2e, CP_3f, DP_1B; that is, P_1 must lie within $P_2P_3P_4$, P_2 within $P_3P_4P_1$, P_3 within $P_4P_1P_2$, or P_4 within $P_1P_2P_3$. What is the chance of P_4 falling within one of these four areas? The chance of a point falling in AP_1E, BP_1D, or CP_1F, is, as we have seen, $\dfrac{\alpha+\beta+\gamma}{2(\alpha+\beta+\gamma)}$. Now conceive EF to move parallel to itself till it assumes the position *ef*. The effect of this transference is that the area AP_1E loses ϵ, and the area CP_1F loses ϕ, while δ is a gain to the favourable chances. Accordingly the chance of P_4 falling within one of the areas $P_1P_2P_3$, AP_2e, CP_3f, DP_1B, is $\dfrac{\alpha+\beta+\gamma-(\epsilon+\phi-\delta)}{2(\alpha+\beta+\gamma)}=\frac{1}{2}-\dfrac{\epsilon+\phi-\delta}{2(\alpha+\beta+\gamma)}$. The ratio $\dfrac{\epsilon+\phi-\delta}{2(\alpha+\beta+\gamma)}$ is $+$; for δ can only become infinite by either P_1P_2 or P_1P_3 or both becoming infinite; and even then $\epsilon+\phi$ is infinitely greater than δ. If any doubt be felt on this point, let it be considered that though P_2P_3 can be made greater than any assignable quantity, it cannot be made infinite by virtue of any position which can be assumed by AB and CD; for when the angle $AP_1C=\pi$, AB and CD coincide by the rotation of one or both those straight lines round P_1: but in this case one of those lines at least does not pass through any point in *ef*, which is contrary to the assumption.

We have then a positive magnitude $\epsilon+\phi-\delta$, which, in order that Mr. WILSON's value $\frac{1}{3}$ may hold good, must $=\frac{1}{3}(\alpha+\beta+\gamma)$; whereas $\dfrac{\alpha+\beta+\gamma}{\epsilon+\phi-\delta}$ may be made greater than 3, 4.... or any assignable magnitude.

1868. (Proposed by J. M. WILSON, M.A.)—Three straight lines are drawn at random on an infinite plane, and a fourth line is drawn at random to intersect them; find the probability of its passing through the triangle formed by the other three.

Solution by PROFESSOR WHITWORTH, M.A.

Of the four lines, two must and two must not pass within the triangle formed by the remaining three.

Since all are drawn at random, the chance that the last drawn should pass through the triangle formed by the other three, is consequently $\frac{1}{2}$.

The only case of exception will be when some of the lines are parallel; but the chance of parallelism is less than any assignable chance.

Hence the result $\frac{1}{2}$ remains correct.

1441. (Proposed by Dr. SALMON, F.R.S.)—A teetotum of n sides is spun an indefinite number of times, and the numbers turning up are added together; what is the chance that a given number s will be actually arrived at?

Solution by THOMAS BOND SPRAGUE, M.A.

A little consideration will show that the probability of s being arrived at in exactly i trials is the coefficient of x^s in $\left(\frac{x+x^2+x^3+\,..\,+x^n}{n}\right)^i$, or in H^i suppose. In the same way, the probability of s being arrived at in $(i+1)$ trials is the coefficient of x^s in H^{i+1}. Thus the total probability of s being arrived at in an indefinite number of trials is the coefficient of x^s in $\mathrm{H}+\mathrm{H}^2+\mathrm{H}^3+\,..\,+\mathrm{H}^i+\,..$, that is in $\frac{\mathrm{H}}{1-\mathrm{H}}$, or in $\frac{1}{1-\mathrm{H}}-1$, which is the same thing as the coefficient of x^s in $\frac{1}{1-\mathrm{H}}$. Substituting for H its value, the probability required is the coefficient of x^s in

$$\frac{n(1-x)}{n(1-x)-x(1-x^n)}, \text{ or in } \frac{n(1-x)}{n-(n+1)x+x^{n+1}}.$$

We must now expand this quantity in powers of x. For this purpose, writing for shortness' sake y for $\left(1-\frac{n+1}{n}x\right)$, we put it in the form

$$(1-x)\left(y+\frac{x^{n+1}}{n}\right)^{-1}=(1-x)\left(y^{-1}-\frac{x^{n+1}}{n}y^{-2}+\frac{x^{2n+2}}{n^2}y^{-3}-\ldots\right).$$

The general term of this series is

$$(-1)^t(1-x)\frac{x^{tn+t}}{n^t}y^{-t-1}, \quad \text{or} \quad (-1)^t(1-x)\frac{x^{tn+t}}{n^t}\left\{1+(t+1)\frac{n+1}{n}x+\ldots\right.$$
$$\left.\ldots+\frac{(t+1)(t+2)\ldots(t+r)}{1.2.3\ldots r}\left(\frac{n+1}{n}\right)^r x^r+\ldots\right\};$$

and the coefficient of x^{tn+t+r} in this expansion is

$$\frac{(-1)^t}{n^t}\left\{\frac{(t+1)(t+2)\ldots(t+r)}{1.2.3\ldots r}\left(\frac{n+1}{n}\right)-\frac{(t+1)(t+2)\ldots(t+r-1)}{1.2.3\ldots(r-1)}\right\}\left(\frac{n+1}{n}\right)^{r-1}$$

$$=\frac{(-1)^t}{n^t}\cdot\frac{1.2.3\ldots(t+r-1)}{1.2\ldots t.1.2\ldots(r-1)}\cdot\frac{tn+t+r}{rn}\left(\frac{n+1}{n}\right)^{r-1}$$

$$=(-1)^t\cdot\frac{(r+1)(r+2)\ldots(r+t-1)}{1.2.3\ldots t}(tn+t+r)\frac{(n+1)^{r-1}}{n^{r+t}}.$$

If now we put $s=tn+t+r$, this is the coefficient of x^s, and is the general term of the probability required. Substituting in the last of the above forms for r its value $s-tn-t$, it becomes

$$(-1)^t\cdot\frac{(s-tn-t+1)(s-tn-t+2)\ldots\ldots(s-tn-1)}{1.2.3\ldots\ldots t}\cdot s\cdot\frac{(n+1)^{s-tn-t-1}}{n^{s-tn}}.$$

We must here give t all the values of which it is capable, viz., 0, 1, 2, .. T, the last being the integer quotient obtained on dividing s by $n+1$.

The required probability is thus found to be

$$\frac{(n+1)^{s-1}}{n^s}-s\cdot\frac{(n+1)^{s-n-2}}{n^{s-n}}+\frac{s(s-2n-1)}{1.2}\cdot\frac{(n+1)^{s-2n-3}}{n^{s-2n}}$$

$$-\frac{s(s-3n-1)(s-3n-2)}{1.2.3}\cdot\frac{(n+1)^{s-3n-4}}{n^{s-3n}}+\ldots.$$

which agrees with the result obtained in a different manner by Mr. WOOLHOUSE, *Reprint*, Vol. I. p. 77.

1480. (Proposed by Professor SYLVESTER.)—Prove that if through the middle point of either diagonal of any of the three quadrilateral faces of a tetrahedral frustum, and the middle points of the two edges which meet but are not in the same face with that diagonal, a plane be drawn, the six planes thus obtained will touch the same cone.

Solution by SAMUEL ROBERTS, M.A.

Complete the tetrahedron, and take as origin the apex so obtained, and as axes the edges meeting at that apex. The coordinates of the six angles of the frustum may be taken to be $x_1, x_2;\ y_1, y_2;\ z_1, z_2$. The plane passing through $(\frac{1}{2}x_2, \frac{1}{2}y_1, 0)$, $(0, \frac{1}{2}y_1, \frac{1}{2}z_1)$, $(\frac{1}{2}x_2, 0, \frac{1}{2}z_2)$ has for its equation

$$2(xy_1z_1+yz_2x_2+zx_2y_1)=x_2y_1(z_1+z_2).$$

We can at once write the equations of two other planes: viz., these are

$$2(xy_2z_2+yz_1x_1+zx_1y_2)=x_1y_2(z_1+z_2),\quad 2(xz_2y_2+yx_1z_2+zx_1y_1)=x_1z_2(y_1+y_2).$$

These planes intersect in the point

$$x=\frac{x_1x_2(y_2z_2-y_1z_1)}{2(\mathrm{K}-\mathrm{L})},\quad y=\frac{y_1y_2(z_2x_2-z_1x_1)}{2(\mathrm{K}-\mathrm{L})},\quad z=\frac{z_1z_2(x_2y_2-x_1y_1)}{2(\mathrm{K}-\mathrm{L})},$$

where K, L are put for $x_2y_2z_2$, $x_1y_1z_1$.

The symmetry of the coordinates shows that the six planes drawn according to the conditions specified intersect in one point.

The planes may, therefore, be represented by

$$y_1z_1\mathrm{X}+x_2z_2\mathrm{Y}+x_2y_1\mathrm{Z}=0\ldots.(1),\qquad y_1z_2\mathrm{X}+x_2z_2\mathrm{Y}+x_1y_1\mathrm{Z}=0\ldots.(4),$$
$$y_2z_2\mathrm{X}+x_1z_1\mathrm{Y}+x_1y_2\mathrm{Z}=0\ldots.(2),\qquad y_1z_1\mathrm{X}+x_2z_1\mathrm{Y}+x_2y_2\mathrm{Z}=0\ldots.(5),$$
$$y_2z_1\mathrm{X}+x_1z_1\mathrm{Y}+x_2y_2\mathrm{Z}=0\ldots.(3),\qquad y_2z_2\mathrm{X}+x_1z_2\mathrm{Y}+x_1y_1\mathrm{Z}=0\ldots.(6);$$

and it is only necessary to show that, considered as Trilinear Coordinates, the coefficients of X, Y, Z represent points on a conic.

Forming an equation to the locus from (1), (2), (3), (4), we get

$$(xx_2-zz_1)(xx_1-zz_2)=p\left\{(xx_1+yy_1-zz_2)\mathrm{K}-(xx_2+yy_2-zz_1)\mathrm{L}\right\}$$
$$\times\left\{(yy_1+zz_1-xx_2)\mathrm{K}-(yy_2+zz_2-xx_1)\mathrm{L}\right\}\ldots\ldots..(a).$$

Now when $(y_1z_1,\ x_2z_1,\ x_2y_2)$, $(y_2z_2,\ x_1z_2,\ x_1y_1)$ are substituted for x, y, z in (a), we have the same results. Therefore the six sets of coefficients represent points on a conic, and reciprocally the six planes envelope a cone.

1481. (Proposed by Professor HIRST.)—Find the envelope of a conic which circumscribes a given triangle, and is cut harmonically by two fixed straight lines.

N.B.—A conic is said to be cut harmonically in two pairs of points, when the tangents at those points cut every other tangent harmonically, or, what amounts to the same thing, when the connectors of those points with any other point on the conic form a harmonic pencil.

I. *Solution by* H. R. GREER, B.A.

Granting that a conic through three fixed points may be represented by a line in such wise that to each point on the conic shall correspond one point on the line, and that the anharmonic ratio of four points on the conic shall be equal to that of the corresponding points on the line, the above problem may be reduced to one of a lower degree, so to speak. And, if this be solved, a retransformation of the result will be necessary in order to obtain the absolute solution of the question proposed.

Now, the possibility of this representation has been established by divers methods of geometrical derivation, and, notably, by that of Quadric Inversion, the fundamental principles of which may be thus summed up,—that being given three fixed points, "principal points," an arbitrary curve of the degree n not passing through any of the principal points will be transformed into one of the degree $2n$, passing n times through each of them; and this transformation (or, rather, representation) will be effected in such a manner that corresponding points shall connect homographically with the principal points. Hence an arbitrary line will be transformed into a conic passing through the three fixed points, anharmonic ratios being preserved. Loci of points and envelopes of curves will be transformed into the loci and envelopes of the corresponding points and curves.

Having recalled these principles to mind, the proposed problem may be replaced by the following:—Find the envelope of a line which is cut harmonically by two fixed conics. This envelope is a conic touching the four

common tangents of the two fixed conics. From which I conclude that the envelope in the proposed Question is a curve of the 4th degree, having double points at the vertices of the given triangle, and touching the four conics which can be described through these three vertices tangential to both the fixed lines.

II. *Solution by* SAMUEL ROBERTS, M.A.

Let a conic of the system be represented by $\frac{A}{\alpha}+\frac{B}{\beta}+\frac{C}{\gamma}=0$, and the two fixed lines by $L\alpha+M\beta+N\gamma=0$, $L'\alpha+M'\beta+N'\gamma=0$; then each of these lines is to pass through the pole of the other; hence we have

$$A^2LL'+B^2MM'+C^2NN'-2AB\,(LM'+L'M)-2BC\,(MN'+M'N)$$
$$-2CA\,(NL'+N'L)=0.$$

The required envelope is obtained by taking, subject to this condition, the envelope of $\frac{A}{\alpha}+\frac{B}{\beta}+\frac{C}{\gamma}=0$, and the result is

$$\frac{1}{\alpha^2}(MN'-M'N)^2+\frac{1}{\beta^2}(NL'-N'L)^2+\frac{1}{\gamma^2}(LM'-L'M)^2$$
$$-\frac{2}{\alpha\beta}(MN'-M'N)\,(NL'-N'L)-\frac{2}{\beta\gamma}(NL'-N'L)\,(LM'-L'M)$$
$$-\frac{2}{\gamma\alpha}(LM'-L'M)\,(MN'-M'N)=0.$$

1864. (Proposed by the Rev. J. BLISSARD.)—Prove that

$$(1)\ldots 1-n+\frac{n(n-1)}{1.2}\ldots\pm\frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}=\pm\frac{(n-1)(n-2)\ldots(n-r)}{1.2\ldots r},$$

$$(2)\ldots\frac{1}{m+1}\ldots+\frac{1}{m+n}=\frac{n(n+1)\ldots(n+m)}{1.2\ldots m}\left\{\frac{1}{(m+1)^2}-\frac{n-1}{1}\cdot\frac{1}{(m+2)^2}+\&c.\right\}.$$

Solution by G. C. DE MORGAN, M.A.

1. Call the quantity on the left-hand side $\phi(r)$; then we have

$$\Delta\,\phi(r)=(-)^{r+1}\cdot\frac{n(n-1)\ldots(n-r)}{1.2\ldots r(r+1)}=(-)^r\cdot\frac{n(n-1)\ldots(n-r)}{1.2\ldots(r-1)}\cdot\Delta\frac{1}{r}.$$

Applying the formula $\Sigma u_r\Delta v_r=u_rv_r-\Sigma v_{r+1}\Delta u_r$, we get

$$\phi(r)=(-)^r\cdot\frac{n(n-1)\ldots(n-r)}{1.2\ldots r}-(n-1)\cdot\phi(r)+\text{a } \textit{constant},$$

therefore $$\phi(r)=(-)^r\cdot\frac{(n-1)(n-2)\ldots(n-r)}{1.2.3\ldots r};$$

the constant being 0, as is found by making $r=1$.

2. The series on the left may be put in the form

$$-(m+1)(m+2)..(m+n)\,.\,\frac{d}{dm}\,.\,\frac{1}{(m+1)(m+2)..(m+n)},$$

or

$$-\frac{(m+1)(m+2)..(m+n)}{1\,.\,2\,..\,(n-1)}\,.\,\frac{d}{dm}\,(-\Delta)^{n-1}\,\frac{1}{m+1}.$$

Developing $(-\Delta)^{n-1}$ in powers of $E = 1+\Delta$, and substituting $\frac{1}{m+2}$ for $E\,.\,\frac{1}{m+1}$, &c., we get

$$-\frac{(m+1)(m+2)..(m+n)}{1\,.\,2\,..(n-1)}\,\frac{d}{dm}\left\{\frac{1}{m+1}-\frac{n-1}{1}\,.\,\frac{1}{m+2}+\&c.\right\}$$

$$=\frac{n(n+1)..(n+m}{1\,.\,2\,.\,3\,..\,m}\left\{\frac{1}{(m+1)^2}-\frac{n-1}{1}\,.\,\frac{1}{(m+2)^2}+\frac{(n-1)(n-2)}{1\,.\,2}\,.\,\frac{1}{(m+3)^2}-\&c.\right\}.$$

1885. (Proposed by the Rev. A. F. TORRY, M.A.)—Investigate the following constructions for determining the point (T) of intersection of the common tangents of an ellipse and its circle of curvature at P. If O be the centre of the circle, C that of the ellipse, S either focus; then (1) T lies on the confocal hyperbola which passes through P; (2) OC bisects PT; and (3) SP, ST are equally inclined to OS.

Solution by the PROPOSER.

1. The first construction is deduced as follows. If P and Q be two points on a conic, the tangents at which meet in T; and if from P and Q tangents be drawn to a confocal conic, these form a quadrilateral in which can be inscribed a circle with centre T: moreover, the extremities of either of the other diagonals of this quadrilateral lie on another confocal conic. This may be proved by showing that the perpendiculars from T upon the sides of this quadrilateral are in a constant ratio to those drawn from T at right angles to the focal distances of P and Q. (See the articles by Mr. C. TAYLOR, in Nos. XI. and XII. of the *Messenger of Mathematics.*)

Now if an ellipse and a confocal hyperbola cut in P, and the tangent to the ellipse at P cut the hyperbo'a again in T; and if from a point on the hyperbola indefinitely near to P, and also from T, tangents be drawn to the ellipse, the inscribed circle of the quadrilateral so formed will be ultimately the circle of curvature at P.

The proposition enumerated above may be otherwise stated thus:—"If a quadrilateral be formed by drawing common tangents to an ellipse and a circle, the extremities of either of its diagonals lie on a confocal conic." Quest. 1814 is a particular case of the same proposition, two of the tangents coinciding in one of the sides of the triangle.

2. The second construction follows from the first if we remember that the centre of curvature is the pole of PT with respect to the confocal hyberbola.

It may also be proved independently. Considering, as before, that the ellipse and circle are inscribed in the same quadrilateral, three of whose sides ultimately coincide with the tangent PT; PT becomes a diagonal of the quadrilateral, and "the centres of all inscribed conics lie on a straight line bisecting the diagonals;" which proves the proposition. It is not difficult to deduce the first construction from the second by pure analysis.

3. The third construction is obtained by reciprocating with respect to either of the foci of the ellipse the well-known theorem, that "the common tangent of an ellipse and its circle of curvature at any point are equally inclined to the major axis of the ellipse."

1890. (Proposed by PROFESSOR CAYLEY.)—Find the equation of a conic passing through three given points and having double contact with a given conic.

Solution by the PROPOSER.

Let the given points be the angles of the triangle ($x=0$, $y=0$, $z=0$), and let the equation of the given conic be $U=(a, b, c, f, g, h)(x, y, z)^2=0$; then the equation of the required conic is

$$U-(x\sqrt{a}+y\sqrt{b}+z\sqrt{c})^2=0,$$

for this is a conic having double contact with the conic $U=0$, and, since the terms in (x^2, y^2, z^2) each vanish, it is also a conic passing through the given points.

It is clear that there are four conics satisfying the conditions of the Problem, viz., putting for shortness

$$P=x\sqrt{a}+y\sqrt{b}+z\sqrt{c},\quad P_1=-x\sqrt{a}+y\sqrt{b}+z\sqrt{c},$$
$$P_2=x\sqrt{a}-y\sqrt{b}+z\sqrt{c},\quad P_3=\ x\sqrt{a}+y\sqrt{b}-z\sqrt{c},$$

the four conics are $U-P^2=0$, $U-P_1^2=0$, $U-P_2^2=0$, $U-P_3^2=0$.

It may be remarked that the conics P, P have a fourth intersection lying on the line $y\sqrt{b}+z\sqrt{c}=0$, and the conics P_2, P_3 a fourth intersection lying on the line $y\sqrt{b}-z\sqrt{c}$; which two lines are harmonics in regard to the lines $y=0$, $z=0$.

Similarly the conics P_1, P_2 have a fourth intersection on the line $x\sqrt{a}+z\sqrt{c}=0$, and the conics P_1, P_3 a fourth intersection on the line $x\sqrt{a}-z\sqrt{c}=0$; which two lines are harmonics in regard to the lines $z=0$, $x=0$. And the conics P_1, P_3 have a fourth intersection on the line $x\sqrt{a}+y\sqrt{b}=0$, and the conics P_2, P_4 a fourth intersection on the line $x\sqrt{a}-y\sqrt{b}=0$; which two lines are harmonics in regard to the lines $x=0$, $y=0$. It may further be remarked that the equations of any two of the four conics may be taken to be

$$\alpha yz+\beta zx+\gamma xy=0,\quad \alpha' yz+\beta' zx+\gamma' xy=0.$$

The general equation of a conic having double contact with each of these conics then is

$$n^2z^2-2n(\gamma\alpha'+\gamma'\alpha)zy-2n(\gamma\beta'+\gamma'\beta)zx-4n\gamma\gamma' xy$$
$$+[(\beta\gamma'-\beta'\gamma)x-(\gamma\alpha'-\gamma'\alpha)y]^2=0.$$

where n is arbitrary: and, having double contact with this conic, we have (besides the above-mentioned two conics) two new conics each passing through the angles of the triangle; viz., writing for greater convenience

$$n = \frac{(\beta\gamma'-\beta'\gamma)(\gamma\alpha'-\gamma'\alpha)}{\mathrm{K}-\gamma\gamma'}, \quad \text{or} \quad \mathrm{K} = \gamma\gamma' + \frac{(\beta\gamma'-\beta'\gamma)(\gamma\alpha'-\gamma'\alpha)}{n},$$

then the equations of the two new conics are

$$\gamma'\alpha\, yz + \gamma\beta'\, zx + \mathrm{K}xy = 0, \quad \gamma\alpha'\, yz + \gamma'\beta\, zx + \mathrm{K}xy = 0.$$

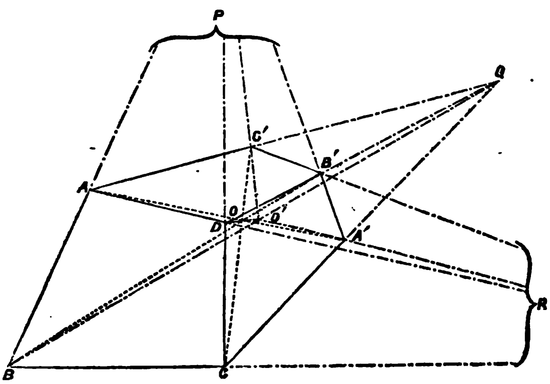
In fact, writing the equation under the form

$$\begin{aligned} &[xz + (\beta\gamma'-\beta'\gamma)\,x + (\gamma\alpha'-\gamma'\alpha)\,y]^2 \\ &-4\,(\beta\gamma'-\beta'\gamma)(\gamma\alpha'-\gamma'\alpha)\,xy - 4n\gamma\gamma' xy \\ &-2n\,(\beta\gamma'-\beta'\gamma)\,xz - 2n\,(\beta\gamma'+\beta'\gamma)\,xz \\ &-2n\,(\gamma\alpha'-\gamma'\alpha)\,yz - 2n\,(\gamma\alpha'+\gamma'\alpha)\,yz = 0, \end{aligned}$$

we at once see that this is a conic having double contact with the conic $\gamma'\alpha\, yz + \gamma\beta' zx + \mathrm{K}xy = 0$, the equation of the chord of contact being $nz + (\beta\gamma'-\beta'\gamma)\,x + (\gamma\alpha'-\gamma'\alpha)\,y = 0$: and similarly it has double contact with the conic $\gamma\alpha' yz + \gamma'\beta zx + \mathrm{K}xy = 0$, the equation of the chord of contact being $nz - (\beta\gamma'-\beta'\gamma)\,x - (\gamma\alpha'-\gamma'\alpha)\,y = 0$.

1893. (Proposed by C. W. Merrifield, F.R.S.)—If the edges of any hexahedron meet four by four in three points, then the four diagonals meet in a point.

Solution by the Proposer.



The construction implies that the hexahedron should be the common frustum of three four-sided pyramids. Now if we consider the diagonal planes of these, that is to say, the planes through the vertices and the dia-

gonals of the bases, it is clear that the four diagonals of the hexahedron lie two by two on these six diagonal planes. Hence they pass through a point.

This demonstration assumes as a lemma a particular case of the following general proposition, which may be established inductively :—"If n lines lie two by two on $\frac{1}{2}n(n-1)$ planes, they all pass through a point." To establish this, begin with two lines in one plane, and add a line at a time. The proposition is then obvious.

II. *Solution by* J. R. ALLEN.

Let ABCDA′B′C′D′ be the given hexahedron, whose edges

BA, CD, A′B′, D′C′ meet in one point P,

AC′, DB′, CA′, BD′ meet in one point Q,

AD, C′B′, D′A′, BC meet in one point R;

it is required to show that the diagonals AA′, BB′, CC′, DD′ intersect in one point.

It is evident by Euc. xi. 2, that

AA′, BB′ intersect in plane PBA′	BB′, CC′ intersect in plane RC′B
AA′, CC′ intersect in plane QAC	BB′, DD′ intersect in plane QDB
AA′, DD′ intersect in plane RAD′	CC′, DD′ intersect in plane PCD′
therefore AA′ cuts BB′, CC′, DD′	CC′ cuts AA′, BB′, DD′
BB′ cuts AA′, CC′, DD′	DD′ cuts AA′, BB′, CC′.

And it is evident that if out of four straight lines every one intersects the three remaining ones, they must all intersect in one point.

III. *Solution by the* REV. R. TOWNSEND, M.A.

By homographic transformation of the figure, the three points of concurrence of the three different quartets of connectors of different pairs of adjacent vertices may be sent to infinity, in which case the four connectors of the four pairs of opposite vertices pass evidently through the centre of the parallelepiped into which the hexahedron becomes then transformed. Or, without any transformation, the four latter lines pass evidently, in every case, through the pole of the plane determined by the three points of concurrence of the three former quartets, with respect to any quadric passing through the eight vertices of the figure.

As a hexahedron reciprocates into an octohedron, the reciprocal property, that when, of an octohedron, the three different quartets of intersections of different pairs of adjacent faces are each complanar, the four intersections of the four pairs of opposite faces are also complanar, appears in the same manner from the reciprocal consideration, that the four latter lines lie evidently, in every case, in the polar plane of the point determined by the three planes of complanarity of the three former quartets, with respect to any quadric touching the eight faces of the figure.

1895. (Proposed by the Rev. R. TOWNSEND, M.A.)—Two circles A and B, whose radii are a and b, touch at two points P and Q a common circle

whose radius is r; show that the length of their common tangent (AB), external or internal according as their contacts with it are of similar or opposite species, is given by the formula $(AB) = \dfrac{\sqrt{(r \pm a)\,.\,(r \pm b)}}{r}\,.\,(PQ)$;
and hence prove immediately the following extension of Ptolemy's Theorem given by Mr. Casey. When four circles A, B, C, D touch a common circle, the six common tangents, AB, &c., of their six groups of two external or internal according as the contacts of the two with the common circle are similar or opposite, are connected by the relation

$$(BC)\,.\,(AD) + (CA)\,.\,(BD) + (AB)\,.\,(CD) = 0.$$

N.B.—By supposing the radius of one of the four circles, D, to be $= 0$, Mr. Casey has obtained immediately from this relation the following equation for a pair of conjugate circles touching the remaining three A, B, C; viz.,

$$(BC)\,.\,\sqrt{(\alpha)} + (CA)\,.\,\sqrt{(\beta)} + (AB)\,.\,\sqrt{(\gamma)} = 0\,;$$

where $\alpha = 0$, $\beta = 0$, $\gamma = 0$ are the equations of A, B, and C.

Solution by A. Renshaw; W. H. Laverty; S. W. Bromfield; J. H. Taylor, B.A.; *and many others.*

1. If A′, B′, are the centres of the two circles, we have

$$\frac{2r^2-(PQ)^2}{2r^2} = \cos O = \frac{(r \pm a)^2+(r \pm b)^2-(A'B')^2}{2\,(r \pm a)\,(r \pm b)},$$

therefore $$\frac{PQ^2}{r^2} = \frac{(A'B')^2-(a \pm b)^2}{(r \pm a)\,(r \pm b)} = \frac{(AB)^2}{(r \pm a)\,(r \pm b)}\,;$$

whence $$r\,.\,(AB) = \sqrt{(r \pm a)\,(r \pm b)}\,.\,(PQ).$$

2. If the radii of the four circles be a, b, c, d, and P, Q, R, S the points of contact, we have, by (1) and Ptolemy's theorem (Euc. VI., D),

$$(BC)\,.\,(AD) + (CA)\,.\,(BD) + (AB)\,.\,(CD) =$$
$$\frac{\sqrt{(r \pm a)\,(r \pm b)\,(r \pm c)\,(r \pm d)}}{r^2}\left\{(QR)\,.\,(PS) + (RP)\,.\,(QS) + (PQ)\,.\,(RS)\right\} = 0.$$

3. When the radius of the circle D vanishes, the points D and S coincide, and may be anywhere on the circumference of the circle (O say) which touches the circles A, B, C: moreover if $\alpha = 0$ be the equation of the circle A, the length of the tangent from any point on O to A will be $\sqrt{(\alpha)}$ (Salmon's *Conics*, Art. 90); hence the relation in (2) becomes

$$(BC)\,.\,\sqrt{(\alpha)} + (CA)\,.\,\sqrt{(\beta)} + (AB)\,.\,\sqrt{(\gamma)} = 0.$$

1901. (Proposed by R. Tucker, M.A.)—Find the curve whose circle of curvature always passes through a fixed point.

I. *Solution by* Professor Mannheim.

Taking the given point as origin, the inverse of the required curve must be a straight line, otherwise it could not have three-pointic contact with all

the straight lines inverse to the circles of curvature of the primitive. This primitive, therefore, must necessarily be itself a circle passing through the given point.

II. *Solution by the* PROPOSER; J. H. TAYLOR, B.A.; *and others.*

If α, β be the coordinates of the centre of curvature, and p, q stand for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, respectively, we have

$$\alpha = x - \frac{p(1+p^2)}{q}, \quad \beta = y + \frac{1+p^2}{q},$$

and the equation to the circle of curvature will be

$$(X-x)^2 + 2(X-x)\frac{p(1+p^2)}{q} + (Y-y)^2 - 2(Y-y)\frac{1+p^2}{q} = 0.$$

(i.) When the circle always passes through a fixed point, taking this point for origin, we have

$$x^2+y^2 = 2x\frac{p(1+p^2)}{q} + 2y\frac{1+p^2}{q};$$

which leads to a circle passing through the given point, and the equation to the circle of curvature is $X^2+Y^2 = 2\alpha X + 2\beta Y$.

(ii.) When the circle always touches a fixed line, take this line for one of the axes, and we have $\alpha=\rho$ or $\beta=\rho$, where ρ is the radius of curvature; and these equations lead to a circle touching the fixed line.

1925. (Proposed by J. GRIFFITHS, M.A.)—Given four points on a circle whose radius is r; show that the centroids (centres of gravity of the areas) of the four triangles that can be formed from them lie on another circle, whose radius is $\frac{1}{3}r$.

Solution by F. D. THOMSON, M.A.; W. H. LAVERTY; R. TUCKER, M.A.; *the* REV. J. L. KITCHIN, M.A.; *and others.*

If A, B, C, D be the four points, and A′, B′, C′, D′ the respective centroids of the triangles BCD, CDA, DAB, ABC, it is clear that the line A′B′ is parallel to AB and equal to $\frac{1}{3}$AB; for A′ and B′ lie respectively on the lines joining B and A with the middle point (M suppose) of CD, and

$$MA' : MB = MB' : MA = 1 : 3.$$

Thus the sides of the quadrilateral A′B′C′D′ are parallel to those of ABCD, and equal to one-third of those sides in linear magnitude; whence the truth of the theorem is obvious.

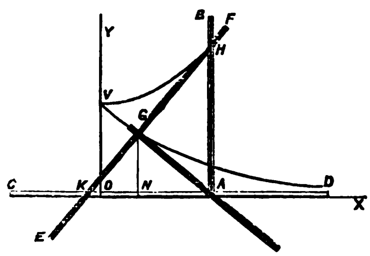
[Mr. TOWNSEND remarks that the property in question is a particular case of the following:—

If A, B, C, D, E, F, &c. be the positions of any number (n) of equal masses distributed in any manner in space, O that of their centre of gravity,

and A′, B′, C′, D′, E′, F′, &c. those of the centres of gravity of their n different groups of $(n-1)$; then always the two systems of n points A, B, C, D, E, F, &c. and A′, B′, C′, D′, E′, F′, &c. are similar, oppositely placed with respect to each other, have O for their centre of similitude, and have $(n-1):1$ for their ratio of similitude. The truth of this is evident, for the several lines AA′, BB′, CC′, DD′, EE′, FF′, &c. all connect through O, and are there divided internally in the common ratio of $(n-1):1$.]

1801. (Proposed by W. H. Laverty.)—A thin bar of wood AB with a groove running along the middle has a perpendicular arm CD projecting each way; and a similar piece EGAF is connected with it by a pivot at A. In EGAF there is a small sharp-edged wheel at G, turning on a pivot which lies in the plane of EGAF, and in the direction of EGF. A pencil is inserted in the concourse of both grooves at H; and CAD being slid along the axis of X, two curves are generated. Find them, and show that the curve described by the pencil is the evolute of that described by the wheel.

Solution by the Proposer.



Here AG $(=a)$, which is constant in length, is a tangent to the lower curve, which is therefore the *Tractory*, whose equation (if $ON=x$, and $NG=y$) is

$$x = -(a^2-y^2)^{\frac{1}{2}} + a\log\frac{a+(a^2-y^2)^{\frac{1}{2}}}{y}.$$

Again, let $OA=x_1$, $AH=y_1$; then

$$x_1 = ON+NA = x+(a^2-y^2)^{\frac{1}{2}} = a\log\frac{a+(a^2-y^2)^{\frac{1}{2}}}{y};$$

but $y = y_1 \cdot \dfrac{KN\,.\,NA}{KA\,.\,AN} = y_1 \cdot \dfrac{y^2}{a^2}$; therefore $y = \dfrac{a^2}{y_1}$;

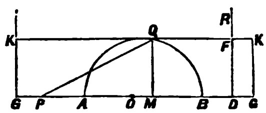
therefore $$x_1 = a\log\frac{y_1+(y_1^2-a^2)^{\frac{1}{2}}}{a}.$$

This is the equation to the *Catenary;* and it is the evolute to the Tractory. See Price's *Calculus*, Vol. I., Fig. 106.

[The construction in the Question seems to have been first given by Sir John Leslie, in his *Geometry of Curve Lines*, p. 398. A fuller description of the instrument is given by Davies (*Hutton's Course*, 12th ed., Vol. II., p. 519), who proposes to call it *Leslie's Catenarian Trammel.*]

1700. (Proposed by T. T. WILKINSON, F.R.A.S.)—The line DR is perpendicular to the diameter AB of a given semicircle AQB; it is required to find in the circumference a point Q such that, if we join Q with P, a point anyhow given in the line DA, and draw QF perpendicular to DR, the sum or difference of PQ and QF may be given.

Solution by the PROPOSER; E. McCORMICK; H. MURPHY; E. FITZGERALD; J. DALE; *and many others.*

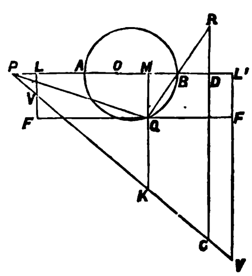


Suppose Q determined; in MP take MG = PQ, and let GK perpendicular to, and QK parallel to, PD meet in K. Now QF = MD by parallels; and MG = PQ; or, PQ + QF = GM + MD = GD = a given line. Hence the point G and the perpendicular GK are given by position. Hence, we have only to draw PQ to a point Q in the circumference of the circle such that PQ = the perpendicular QK, and this has already been done in Quest. 219 of the *Key* by various correspondents.

In like manner, when the *difference* is given: take DG on the contrary side of D, equal to the given difference; draw the perpendicular GK; and PQ, making PQ = QK or MG, as in the above. Hence this is evidently also a case of Question 219, as before.

1764. (Proposed by T. T. WILKINSON, F.R.A.S.)—The same being supposed as in Quest. 1700; it is required to determine Q, so that if QR be drawn making any given angle with DR, the sum, or difference, of PQ and QR may be given.

Solution by the PROPOSER; E. McCORMICK; H. MURPHY; E. FITZGERALD; J. DALE; *and many others.*



Draw DC perpendicular to DP; also PC making ∠ DCP = the given angle. In CP, towards P, take CV = the given sum; draw LVF perpendicular to PD. Now determine the point Q, by the construction in Question 219 of the *Key*, making PQ : QF (a perpendicular upon LV) = PC : PD; the required point is Q.

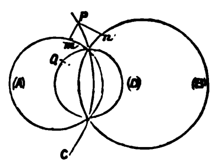
Draw QR making the given angle with DR; and let QM, a perpendicular to PD, meet CP at K. Then, because of equal angles and parallels, DP : PC = DM : QR = DM : CK; whence QR = CK. Also, PQ = QF = PQ : ML = PC : PD = VK : LM; whence PQ = VK. Therefore PQ + QR = VK + KC = VC = the given *sum*.

When the *difference* is given; take CV, on the contrary side of C, equal

to that difference; and draw PQ and QF as before. Then, in the demonstration, it follows that $KC = QR$, $KV = QP$; and therefore $PQ - QR = KV - KC = CV =$ the given difference.

1655. (Proposed by M. W. Crofton, B.A.)—Let the equations of two circles (A) and (B), whose radii are r and r', be $\Theta = 0$ and $\Theta' = 0$; then the two circles (C) and (D), whose equations are $\dfrac{\Theta}{r} - \dfrac{\Theta'}{r'} = 0$ and $\dfrac{\Theta}{r} + \dfrac{\Theta'}{r'} = 0$, intersect at right angles.

Solution by the Proposer.



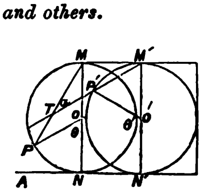
Take a point P on (C), infinitely near the intersection; then it is known that putting the coordinates of P for (x, y) in Θ and Θ' we have $\Theta = 2r \, . \, Pm$, $\Theta' = 2r' \, . \, Pn$; hence $Pm = Pn$, so that the circle (C) bisects the angle of $\Theta = 0$, $\Theta' = 0$.

Also if we took a point Q on (D), infinitely near the intersection, we should have $-\Theta = 2r \, . \, \delta$, $\Theta' = 2r' \, . \, \delta'$, where δ and δ' are the evanescent perpendiculars from Q on (A) and (B); hence the circle (D) bisects the other angle of $\Theta = 0$, $\Theta' = 0$.

Therefore the circles (C) and (D) cut at right angles.

1758. (Proposed by M. W. Crofton, B.A.)—If two tangents to a cycloid include a constant angle, show that their sum has a constant ratio to the included arc of the curve.

Solution by J. Dale; E. McCormick; *and others.*



Let NPM, N'P'M' be two positions of the generating circle corresponding to the points P, P' on the cycloid; and let θ, θ' be the auxiliary angles; then PM, P'M' (intersecting in T) are tangents at P and P', and the included arc of cycloid $= 2(PM - P'M')$. Also, $\alpha = \angle \, MTM' = \frac{1}{2}(\theta' - \theta)$, $MM' = NN' = a(\theta' - \theta)$, and in the triangle MTM',

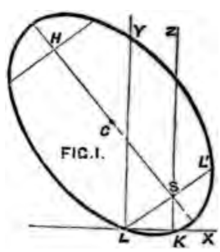
we have $TM = MM' \, . \sin MM'T \, . \operatorname{cosec} MTM' = a \operatorname{cosec} \alpha \, . \, P'M'$;

therefore $TP = PM - a \operatorname{cosec} \alpha \, . \, P'M'$, so also $TP' = a \operatorname{cosec} \alpha \, . \, PM - P'M'$;

therefore $TP + TP' = (1 + a \operatorname{cosec} \alpha)(PM - P'M') = (1 + a \operatorname{cosec} \alpha) \text{ arc } PP'$.

1583. (Proposed by R. Tucker, M.A.) — A system of similar ellipses passes through a fixed point which coincides with one end of their parameters; find the locus of their centres, the envelope of the other parameters, and also that of the system when the locus of the focus is (1) a straight line, (2) a circle.

Solution by the Proposer.



Take L (Fig. 1) for fixed extremity of parameter, and for axis in (1), LX at right angles to the given line KZ, and LY parallel to it. Suppose LK $= k$, e the given eccentricity and (α, β) the coordinates of the centre; then if $\angle$SLK $= \theta$, we have

$$\alpha = k - ae\sin\theta,$$
$$\beta = k\tan\theta + ae\cos\theta,$$
$$k = a(1-e^2)\cos\theta;$$

whence eliminating a and θ, we have

$$(1-e^2)^2\alpha + e(1-e^2)\beta = k(1-e^2+e^4) \text{.. (A)};$$

the equation to a straight line cutting KZ at a distance $ke(1-e^2)^{-1}$ from K equal to half the projection of the line between the foci on KZ.

Now the equation to the parameter through H, a point whose coordinates are $(k - 2ae\sin\theta,\ k\tan\theta + 2ae\cos\theta)$ is

$$y - x\tan\theta = \frac{2ke}{1-e^2}\sec^2\theta,$$

the envelope of which is

$$x^2 = \frac{8ke}{1-e^2}\left(-y + \frac{2ke}{1-e^2}\right) \text{..................... (B)},$$

indicating a parabola whose focus is L, axis YL, and parameter $\dfrac{8ke}{1-e^2}$.

If (r, ϕ) be any point on the ellipse, we have

$$r\left\{1 - e^2\sin^2(\phi-\theta)\right\} = 2a(1-e^2)\left\{\cos(\phi-\theta) + e\sin(\phi-\theta)\right\}$$
$$= 2k\left\{\cos\phi + e\sin\phi + \tan\theta(\sin\phi - e\cos\phi)\right\}.$$

Differentiating and arranging the equations, we have

$$\tan^3\theta\left\{2k(\sin\phi - e\cos\phi)\right\} + \tan^2\theta\left\{2k(\cos\phi + e\sin\phi) - r + e^2r\cos^2\phi\right\}$$
$$+ \tan\theta\left\{2k(\sin\phi - e\cos\phi) - 2e^2r\sin\phi\cos\phi\right\} + 2k(\cos\phi + e\sin\phi)$$
$$- r(1 - e^2\sin^2\phi) = 0,$$

$$\tan^2\theta(e^2r\sin\phi\cos\phi) + e^2r\tan\theta\cos 2\phi + k(\sin\phi - e\cos\phi) - e^2r\sin\phi\cos\phi = 0;$$

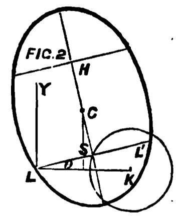
i. e., we have to eliminate $z\ (= \tan\theta)$ between

$$az^3 + bz^2 + 2cz + d = 0, \quad a'z^2 + b'z + c = 0,$$

where $2(a'+c) = a$, and $b - b' = d$; whence we get

$$(b-b')^2a'^3 - 2c(b-b')(b-2b')a'^2 + \left[b^2c^2 + b^2(b-b')\left\{b'(b-2b') + 2c^2\right\}\right]a'$$
$$+ 2c\left\{c^2(2b'^2 - bb' + 2c^2) - b'^3(b-b')\right\} = 0 \text{.............. (C)},$$

where the letters have the above values, and the resulting curve is the required envelope.



Next (Fig. 2) let K be the centre of the given circle, and the axes LK and a line LY at right angles to it.

Then, if c = the given radius, we have

$$\alpha = a(1-e^2)\cos\theta - ae\sin\theta,$$

$$\beta = ae\cos\theta + a(1-e^2)\sin\theta,$$

$$c^2 = a^2(1-e^2)^2 + k^2 - 2ak(1-e^2)\cos\theta;$$

whence, eliminating a and θ, we obtain

$$(\alpha^2+\beta^2)(1-e^2)^2 - 2k(1-e^2)^2\alpha - 2k(1-e^2)e\beta$$
$$= (c^2-k^2)(1-e^2+e^4)\ \ldots\ldots\ (\mathrm{D}),$$

or the locus of centres is a circle.

The equation to the parameter through the point H, whose coordinates are $\left\{a(1-e^2)\cos\theta - 2ae\sin\theta\right\}$, $\left\{2ae\cos\theta + a(1-e^2)\sin\theta\right\}$, is

$$y - x\tan\theta = 2ae\sec\theta = \frac{2ek}{1-e^2} \pm \frac{2e\sec\theta}{1-e^2}(c^2 - k^2\sin^2\theta)^{\frac{1}{2}};$$

whence $\quad (c^2-k^2)\left\{y - \frac{2ek}{1-e^2}\right\}^2\left\{x^2(1-e^2)^2 - 4e^2(c^2-k^2)\right\}$

$$= c^2\left\{x^2(1-e^2) + 4e^2(c^2-k^2)\right\}^2\ \ldots\ldots\ldots\ (\mathrm{E})$$

is the equation to the envelope.

In the particular case when L is on the given circle (or $c=k$), we have

$$y - x\tan\theta = 2ae\sec\theta = \frac{4ke}{1-e^2}\ \ldots\ldots\ldots\ldots\ldots\ldots\ (\mathrm{E}'),$$

showing that the parameters pass through a fixed point on the tangent at L.

The equation to the ellipse in the general case leads to a complicated result, but in the above case it becomes

$$r\left\{1 - e^2\sin^2(\phi-\theta)\right\} = 4k\left\{\cos(\phi-\theta) + e\sin(\phi-\theta)\right\}\cos\theta$$
$$= 2k\left\{\cos\phi + \cos(\phi-2\theta) + e\sin\phi + e\sin(\phi-2\theta)\right\}.$$

Differentiating, we obtain

$$re^2\sin(2\phi-2\theta) = 4k\left\{\sin(\phi-2\theta) - e\cos(\phi-2\theta)\right\};$$

and as a final result we obtain for the equation to the envelope sought

$$(1-e^2)r^4 - 4kr^2\left\{r\cos\theta(1-e^2) + er\sin\theta + k(1+e^2)\right\}$$
$$+ 4k^2(r\cos\theta + er\sin\theta)^2 = 0\ \ldots\ldots\ldots\ldots\ldots\ldots\ (\mathrm{F}).$$

Similar results hold in each case for the curves below the line LK.

On Approximation to a Curvilinear Area.

By Professor De Morgan.

The brevity of the following makes a problem of it. If y_0, y_1, &c., be ordinates of which the abscissæ 0, h, $2h$, &c., differ by h, and if $\mathrm{E}\phi(x) = \phi(x+1)$, we know that $\int_0^{nh} y_x dx$ is $h(\log \mathrm{E})^{-1}$ applied to $y_n - y_0$. If we

can find a function $h\psi E.(E^k-1)^{-1}$, in which, when $E=1+\Delta$, a number of terms of the development in powers of Δ agree with those of $h(\log E)^{-1}$, we may from it construct an approximation to the integral, and find an approximation to the error committed. But, to preserve the symmetry which appears in the two modes of representation of $(\log E)^{-1}$, $\psi E.(E^k-1)^{-1}$, or $\chi\Delta$, should become $-\chi(-z)$ when $z(1-z)^{-1}$ is written for z.

The function $\frac{1}{2}(E+1)(E-1)^{-1}$ satisfies the conditions, and, applied to y_n-y_0, it gives $\frac{1}{2}y_0+y_1+\ldots\ldots+y_{n-1}+\frac{1}{2}y_n$, the *surveyor's term*, as I always call it, of the ordinary method of quadratures. But the surveyors, or at least the engineers, have advanced a step, and have occasion to use *Simpson's Rule*. It is got by making $2n$ subdivisions, and considering the area from y_0 to y_2, from y_2 to y_4, &c., as area of parabolas with axes parallel to y. It is the next case of the method; and ψE is E^2+4E+1.

If P_k abbreviate $\Delta^k y_{2n-k}+(-1)^k\Delta^k y_0$, the development of $\chi\Delta$ is

$$\left(\tfrac{1}{2}y_0+\ldots\ldots+\tfrac{1}{2}y_{2n}\right)-\tfrac{1}{12}P_1-\tfrac{1}{24}P_2-\tfrac{1}{48}P_3-\tfrac{1}{96}P_4-\ldots\ldots$$

The development of $(\log E)^{-1}(y_{2n}-y_0)$ has $-\frac{19}{720}P_3-\frac{3}{160}P_4$.
And $h\psi E(E^2-1)^{-1}(y_{2n}-y_0)$ gives

$$\tfrac{1}{3}h\left(y_0+4y_1+2y_2+4y_3+2y_4+\ldots\ldots+4y_{2n-1}+y_{2n}\right),$$

which is Simpson's rule. It may be corrected by subtracting

$$h\left(\frac{19}{720}-\frac{1}{48}\right)P_3+h\left(\frac{3}{160}-\frac{1}{96}\right)P_4,$$

or

$$\frac{h}{60}\left\{\frac{\Delta^3 y_{2n-3}-\Delta^3 y_0}{3}+\frac{\Delta^4 y_{2n-4}+\Delta^4 y_0}{2}\right\}.$$

The functions ψ will be best found by drawing the parabolas $a+bx$, $a+bx+cx^2$, &c., through two, three, &c. consecutive points. The corrections may be best investigated, I think, in the preceding way. But as the error of Simpson's rule is of the fourth order, I suppose the engineers will not want a biquadratic parabola for a few years to come.

The coefficients of $\left\{\log(1+\Delta)\right\}^{-1}$, so far as usually given, are

$$1,\ \frac{1}{2},\ -\frac{1}{12},\ \frac{1}{24},\ -\frac{19}{720},\ \frac{3}{160},\ -\frac{863}{60480}.$$

The three next, whether ever before printed I know not, are

$$\frac{275}{24192},\ -\frac{33953}{3628800},\ \frac{8183}{1036800}.$$

1521. (Proposed by J. M. WILSON, M.A., F.G.S.)—Show that, in a geometric progression of an odd number of terms, the arithmetic mean of the odd numbered terms is greater than the arithmetic mean of the even-numbered terms, if the common ratio be any positive rational quantity not equal to unity.

Solutions by C. M. INGLEBY, LL.D.

Let the given series be $1+a+a^2+\ldots\ldots+a^{2n}$, and suppose throughout that a is not equal to unity. Then we have to show that

$$\frac{1+a^2+a^4+\ldots\ldots+a^{2n}}{a+a^3+\ldots\ldots+a^{2n-1}}>\frac{n+1}{n}.\quad\text{Write this } \alpha_n>\frac{n+1}{n};\ \text{then}$$

$$\alpha_{n+1}+\frac{1}{\alpha_n}=\frac{1+a^2+a^4+\ldots\ldots+a^{2n+2}}{a+a^3+\ldots\ldots+a^{2n+1}}+\frac{a+a^3+\ldots+a^{2n-1}}{1+a^2+a^4+\ldots+a^{2n}}$$

$$=\frac{1+2(a^2+a^4+\ldots\ldots+a^{2n})+a^{2n+2}}{a+a^3+\ldots\ldots+a^{2n+1}}$$

$$=\frac{(1+a^2+a^4+\ldots\ldots+a^{2n})(1+a^2)}{(1+a^2+a^4+\ldots\ldots+a^{2n})\,a}=a+\frac{1}{a}\ \ldots\ldots\ldots\ldots(1).$$

$$\text{Now } a+\frac{1}{a}>2,\quad\text{that is, }>\frac{n+2}{n+1}+\frac{n}{n+1}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$\text{therefore}\quad \alpha_{n+1}=\left(a+\frac{1}{a}\right)-\frac{1}{\alpha_n}>\frac{n+2}{n+1}+\left(\frac{n}{n+1}-\frac{1}{\alpha_n}\right)\ \ldots\ldots\ldots(3).$$

But if $\alpha_n>\frac{n+1}{n}$, $\frac{n}{n+1}>\frac{1}{\alpha_n}$, and $\left(\frac{n}{n+1}-\frac{1}{\alpha_n}\right)$ is positive;

therefore, by (3), $\quad \alpha_{n+1}>\frac{n+2}{n+1}$, if $\alpha_n>\frac{n+1}{n}$.

But $\alpha_1=a+\frac{1}{a}>\frac{2}{1}$; $\therefore\ \alpha_2>\frac{3}{2}$, $\alpha_3>\frac{4}{3}$, &c., and generally $\alpha_n>\frac{n+1}{n}$.

2. *Otherwise:* Let $\alpha_n=\frac{1+a^2+a^4+\ldots\ldots+a^{2n}}{a+a^3+\ldots+a^{2n-1}}=\frac{a^{2n+2}-1}{a^{2n+1}-a}$;

then, converting this into a continued fraction, we get

$$\alpha_n=\left(a+\frac{1}{a}\right)-\cfrac{1}{\left(a+\frac{1}{a}\right)-\cfrac{1}{\left(a+\frac{1}{a}\right)-\text{\&c.}}}\ \ldots\ldots\ldots\ldots(4).$$

By simple inspection of (4) we obtain the relation (1), viz., $\alpha_{n+1}=a+\frac{1}{a}-\frac{1}{\alpha_n}$, and the rest of the proof follows as before.

[Other Solutions are given in the *Reprint*, Vol. III., p. 15.]

1554. (Proposed by Professor CAYLEY.)—Show that, in the ellipse and its circles of maximum and minimum curvature respectively, the semi-ordinates through the focus of the ellipse are

For the circle of maximum curvature.... $y_1 = a(1-e)(1+2e)^{\frac{1}{2}}$,

for the ellipse........................ $y_2 = a(1-e^2)$,

for the circle of minimum curvature .... $y_3 = \dfrac{a\left\{(1-e^2+e^4)^{\frac{1}{2}}-e^2\right\}}{(1-e^2)^{\frac{1}{2}}}$,

and that these values are in the order of increasing magnitude.

Solution by the Rev. J. L. Kitchin, M.A.; E. McCormick; *and others.*

Let $b^2x^2+a^2y^2 = a^2b^2$ be the equation to the ellipse; then, if ρ be the radius of curvature at any point, we easily get, by substitution in the usual formula, $a^2b^2\rho^2 = (a^2-e^2x^2)^3$. Now ρ is a minimum (ρ_1, say) when $x=a$, and a maximum (ρ_2, say) when $x=0$;

therefore $\rho_1 = a(1-e^2)$, and $\rho_2 = \dfrac{a^2}{b} = \dfrac{a}{(1-e^2)^{\frac{1}{2}}}$.

The equations of the circles corresponding to ρ_1, ρ_2, that is to say, of the circles of maximum and minimum curvature, are respectively

$$(x-ae^2)^2+y^2 = a^2(1-e^2)^2, \quad x^2+\left(y+\frac{ae^2}{(1-e^2)^{\frac{1}{2}}}\right)^2 = \frac{a^2}{1-e^2}.$$

If in the equations of these circles, and in that of the ellipse, we put $x=ae$, we readily obtain the values of the ordinates given in the Question.

Now $y_1 = a(1-e)(1+2e)^{\frac{1}{2}}$, and $y_2 = a(1-e)(1+2e+e^2)^{\frac{1}{2}}$; $\therefore\ y_2>y_1$.

Again, $y_3>y_2$, if $\left\{(1-e^2+e^4)^{\frac{1}{2}}-e^2\right\}^2 > (1-e^2)^3$, or if

$$1-e^2+e^4-2(1-e^2+e^4)^{\frac{1}{2}}+1 > 0, \text{ or if } \left\{(1-e^2+e^4)^{\frac{1}{2}}+1\right\}^2 > 0;$$

hence $y_3>y_2$, and therefore $y_3>y_2>y_1$.

1820. (Proposed by the Rev. J. Blissard.)—Prove that

$$\frac{m^m}{x+m}-\frac{m}{1}\cdot\frac{(m-1)}{x+m-1}+\frac{m(m-1)}{1.2}\cdot\frac{(m-2)^m}{x+m-2}-\&c. = \frac{1.2\ldots.m.x^{m-1}}{(x+1)(x+2)\ldots(x+m)}.$$

Solution by Samuel Roberts, M.A.

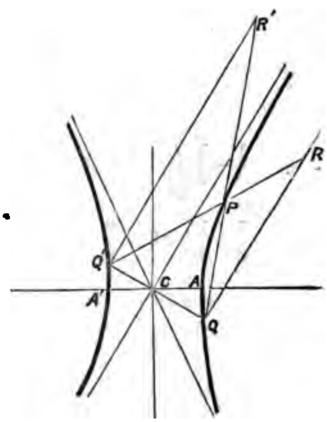
This is the direct result of applying the usual formula for the decomposition of rational Fractions to the right-hand member.

1836. (Proposed by R. Tucker, M.A.)—Through the extremities of a diameter of an hyperbola (or its conjugate) at right angles to one asymptote,

straight lines are drawn parallel to the other; if the straight lines joining the extremities of the diameter to any point on the curve be produced, they will intercept on the parallels portions whose difference is constant.

Solution by the PROPOSER; E. McCORMICK; *and others.*

Let A, A′ be the vertices of the hyperbola, and QCQ′ a diameter at right angles to one of the asymptotes. Connect P, any point on the curve, with Q, Q′; and produce QP, Q′P to meet the parallels to the asymptote (through Q′, Q) in R′, R; then shall the difference between Q′R′ and QR be constant.



For let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the equation to the hyperbola referred to the axes; then, transferring the axes of reference to CQ and the asymptote, and observing that $\tan\theta = \tan ACQ = \frac{a}{b}$, the equation becomes

$$\frac{4cxy}{c^2 - x^2} = \frac{2ab}{c},$$

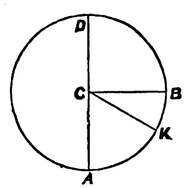
where $c = CQ = \frac{b}{\sqrt{(e^2 - 2)}}$.

Hence, since the difference of Q′R′ and QR $= \frac{4cxy}{c^2 - x^2}$, the property is proved.

1848. (Proposed by M. W. CROFTON, B.A.)—Supposing the density of the population of the metropolitan area (radius 8 miles) to vary inversely as the distance from the centre, find the probability of two persons taken at random living nearer than 8 miles to each other.

Solution by the PROPOSER.

If an indefinite number of radii diverge at equal angular intervals from the centre of a circle, and a series of points are uniformly distributed along these radii, it is clear that the density of these points varies inversely as the distance from the centre.



It is easily seen that the required probability is unaltered by restricting one of the points to the fixed radius AC; we may also restrict the other to the *semicircle* ABD.

Make the angle ACK $= \frac{1}{3}\pi$; and draw CB perpendicular to AD. From

the solution to Quest. 1838 (*Reprint*, Vol. V., p. 53) we see that if the second point is on a radius within the sector ACK, the probability $p=1$; if within KCB, $p=\frac{3\theta-\pi}{2\sin\theta}+2\cos\theta$; if within BCD, $p=\frac{\pi-\theta}{2\sin\theta}$, where θ is the inclination of the radius to CA. Hence the required probability is

$$p=\frac{1}{3}+\frac{1}{\pi}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\left(\frac{3\theta-\pi}{2\sin\theta}+2\cos\theta\right)d\theta+\frac{1}{\pi}\int_{\frac{1}{2}\pi}^{\pi}\frac{\pi-\theta}{2\sin\theta}\,d\theta.$$

But
$$\int_{\frac{1}{2}\pi}^{\pi}\frac{\pi-\theta}{\sin\theta}\,d\theta=\int_{0}^{\frac{1}{2}\pi}\frac{\theta\,d\theta}{\sin\theta},$$

$$\therefore\ p=\frac{1}{3}-\frac{1}{2}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\frac{d\theta}{\sin\theta}+\frac{2}{\pi}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\cos\theta\,d\theta+\frac{3}{2\pi}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\frac{\theta\,d\theta}{\sin\theta}+\frac{1}{2\pi}\int_{0}^{\frac{1}{2}\pi}\frac{\theta\,d\theta}{\sin\theta};$$

$$\text{or, } p=\frac{1}{3}-\frac{1}{4}\log 3+\frac{2}{\pi}\left(1-\frac{\sqrt{3}}{2}\right)+\frac{1}{2\pi}\left(\int_{0}^{\frac{1}{2}\pi}\frac{\theta\,d\theta}{\sin\theta}+3\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\frac{\theta\,d\theta}{\sin\theta}\right).$$

To find series for the above definite integrals, let $t=\tan\frac{1}{2}\theta$; then

$$\int\frac{\theta\,d\theta}{\sin\theta}=2\int\frac{dt}{t}\tan^{-1}t=2\left(t-\frac{t^3}{3^2}+\frac{t^5}{5^2}-\frac{t^7}{7^2}+\&c.\right);$$

therefore $\displaystyle\int_{0}^{\frac{1}{2}\pi}\frac{\theta\,d\theta}{\sin\theta}=2\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\&c.\right)$; also

$$\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\frac{\theta\,d\theta}{\sin\theta}=2\left(1-\frac{1}{3^2}+\&c.\right)-\frac{2}{\sqrt{3}}\left(1-\frac{1}{3^2}\cdot\frac{1}{3}+\frac{1}{5^2}\cdot\frac{1}{3^2}-\&c.\right).$$

The series $1-\frac{1}{3^2}+\frac{1}{5^2}-\&c.$ converges very slowly, but by means of certain artifices, which we have not space to give here, its value may be found without trouble to be ·9159; also

$$1-\frac{1}{3^2}\cdot\frac{1}{3}+\frac{1}{5^2}\cdot\frac{1}{3^2}-\frac{1}{7^2}\cdot\frac{1}{3^3}+\&c.=\cdot 96677.$$

Hence

$$p=\frac{1}{3}-\frac{1}{4}\log 3+\frac{2}{\pi}\left(1-\frac{\sqrt{3}}{2}\right)+\frac{1}{\pi}\left(\cdot 9159+3\times\cdot 9159-\cdot 96677\times\sqrt{3}\right),$$

which will give for the required probability $p=\cdot 7771$, nearly.

1957. (Proposed by the Rev. R. TOWNSEND, M.A.)—Show that the chords of quickest and slowest descent from the highest point of an ellipse in a vertical plane are at right angles to each other and parallel to the axes of the curve.

I. *Solution by* ARTHUR COHEN, B.A.; E. McCORMICK; *and others.*

Take the normal and tangent at the highest point P of the ellipse as the axes of x and y respectively. Then the equation to the ellipse is

$$ax^2 + bxy + cy^2 + dx = 0.$$

Let r be the length of a chord through P making an angle θ with the normal; then putting $r\cos\theta$ for x, and $r\sin\theta$ for y, we have

$$r(a\cos^2\theta + b\cos\theta\sin\theta + c\sin^2\theta) + d\cos\theta = 0.$$

Therefore $\dfrac{r}{\cos\theta}$, which evidently is proportional to the square of the time of descent down the chord, equals $\dfrac{-d}{a\cos^2\theta + b\cos\theta\sin\theta + c\sin^2\theta}$.

Now if we transfer the origin to the centre, the equation to the ellipse becomes

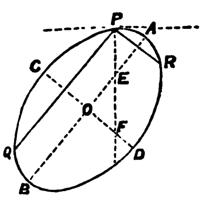
$$ax^2 + bxy + cy^2 + f = 0;$$

and if we denote by ρ the semi-diameter making an angle θ with the normal at P, we have, by putting $\rho\cos\theta$, $\rho\sin\theta$, for x and y,

$$\rho^2(a\cos^2\theta + b\cos\theta\sin\theta + c\sin^2\theta) + f = 0.$$

Hence it follows that $\dfrac{r}{\cos\theta} = \dfrac{d}{f}\rho^2$, and therefore $\dfrac{r}{\cos\theta}$ is a maximum and a minimum when ρ is a maximum and a minimum, that is, when ρ is the major axis and the minor axis; hence the time of descent is a maximum and minimum when the chord r is parallel to the axes of the ellipse.

II. *Solution by* ARCHER STANLEY; M. COLLINS, B.A.; H. MURPHY; *the* REV. J. L. KITCHIN, M.A.; T. J. SANDERSON, B.A.; J. DALE; *the* PROPOSER; *and many others.*



Let the vertical through the highest point P of the ellipse cut the axes AB, CD in E and F. Now PR being drawn perpendicular to AB, a circle with centre E and radius PE will lie wholly *within* the ellipse and touch it at R as well as at P, since the curve is symmetrical about AB. But this being the case, PR is obviously the chord of quickest descent. Similarly the circle whose centre is F and radius PF will not only touch the ellipse in P, but also in Q, the image of P relative to CD; and since this circle lies wholly *without* the ellipse, PQ is the chord of slowest descent.

In the same manner evidently it may be shown that the chords of swiftest and slowest descent from the highest point of an ellipsoid anyhow situated, are parallel respectively to the least and greatest axes of the surface.

1965. (Proposed by H. R. GREER, B.A.)—Four conics through four points form a harmonic system; prove that if two conjugates be a circle and an equilateral hyperbola, the other two must be of equal eccentricities.

Solution by ARCHER STANLEY.

Let S and Σ be the conjugate conics which pass through the intersections of the equilateral hyperbola H with the circle C, and form with these a harmonic system; that is to say, A being any one of the four common intersections, let the tangents at A to S and Σ be harmonic conjugates relative to the tangents at A to H and C.

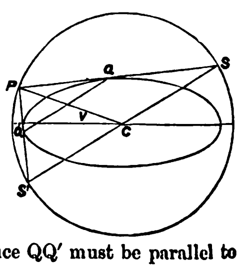
Now, by a well known theorem, the pairs of rays are in involution which are drawn through A parallel to the asymptotes of the several conics passing through the intersections of H and C; and the chords of the arcs which these pairs intercept on C are concurrent; they form, in fact, a pencil of rays homographic with the pencil of tangents at A.

Hence h, c, s, σ, the chords of the arcs of C intercepted by parallels through A to the asymptotes of H, C, S, Σ, form a harmonic pencil. But h is manifestly a diameter of C, and c is at infinity, consequently s and σ are parallel to and equidistant from h. The chords s and σ, therefore, subtend supplemental angles at A, that is to say, the asymptotes of S are inclined to each other at the same angles as are the asymptotes of Σ; which proves the theorem.

N.B.—The system of conics includes two parabolas, which likewise form, with H and C, a harmonic system; their axes are parallel to the connectors of A with the extremities of the diameter perpendicular to h, and consequently are at right angles to each other and inclined at an angle of 45° to the asymptotes of H. They are likewise parallel to the asymptotes of the rectangular hyperbola which is the locus of the centres of all the conics of the system.

1968. (Proposed by A. RENSHAW.)—If from any point P in a circle concentric with a given ellipse, and the radius of which is equal to the distance between the ends of the major and minor axes, a pair of tangents be drawn to the ellipse and produced to meet the circle in the points S and S′; prove that the line SS′ is parallel to the polar of P.

Solution by T. J. SANDERSON, B.A.; *the* REV. R. H. WRIGHT, M.A.; H. TOMLINSON; *the* REV. J. L. KITCHIN, M.A.; *the* PROPOSER; *and others.*

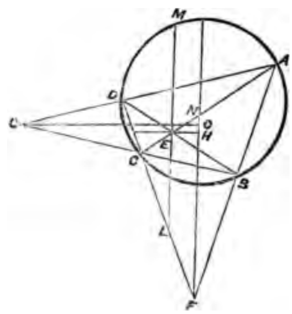


By a well known theorem, the circle concentric with a given ellipse, and of radius equal to the distance between the ends of the major and minor axes, is the locus of the intersection of pairs of tangents to the ellipse which cut at right angles. Hence SPS′ is a right angle; therefore SS′ is a diameter of the circle, and consequently passes through C the centre of the ellipse.

Let QQ′ be the polar of P, and join PC meeting QQ′ in V. Then QQ′ is bisected in V and SS′ in C by the same straight line PC. Hence QQ′ must be parallel to SS′.

1950. (Proposed by Professor SYLVESTER.)—If A, B, C, D be four points in a circle; and if AB, CD produced meet in F, and AD, BC produced meet in G, prove that the lines which bisect the angles F and G are at right angles to each other.

Solution by J. McDOWELL, F.R.A.S.; W. L. BOWDITCH; C. LAW; H. MURPHY; A. COHEN; S. W. BROMFIELD; H. TOMLINSON; W. M. MADDEN; J. DALE; T. J. SANDERSON, B.A.; Rev. R. HARLEY, F.R.S.; M. COLLINS, B.A.; T. COTTERILL, M.A.; *and many others.*



This may be at once deduced from the following theorem, which was set as a question at Christ's College, Cambridge, in 1859.

"ABCD is a quadrilateral inscribed in a circle; AC, BD meet in E; AB, CD produced meet in F; and AD, BC in G; show that the lines bisecting the angles AED, AFD are parallel to one another, and also those bisecting the angles AEB, AGB."

Let FN bisect the angle AFD and meet AC in N, GO bisect the angle AGB and meet FN in O, EH bisect AEB and LM bisect AED.

Since the angles ABD and ACD are equal, therefore the angles FBE and FCE are equal; therefore FCE, CEL and half AFD are together equal to two right angles, and therefore also to FCE, CEL and CLE. Therefore CLE equals CFN, and therefore LM and FN are parallel. Similarly EH and GO are parallel. Therefore EO is a parallelogram, and its opposite angles GOF and MEH are equal, but MEH is obviously a right angle, therefore GOF is a right angle.

II. *Solution by* H. McCOLL.

Before applying the principles of Angular and Linear Notation to prove Professor Sylvester's theorem, I will make a few additions to my former article on the subject. (*Reprint,* Vol. V., p. 74.)

DEF.—Let Z represent any curve, and p any point in the same plane; then z will denote the perpendicular from p upon the curve Z. Interpretation:—Let any tangent touch the curve at the point t; then z denotes the length of the line pt *provided that* pt *is perpendicular to the tangent.* The perpendicular z is understood to be *positive* when p and the point of reference are *on the same side* of the curve, and *negative* when *on opposite* sides of it.

1. In accordance with this definition and previous conventions, whatever be the position of the point of reference we shall have as the equation to the circle

$$(z) = (x-x')^2 + (y-y')^2 + 2(x-x')(y-y')\cos \mathrm{xy} - (z' \sin \mathrm{xy})^2,$$

in which X and Y are any straight lines not parallel, Z the circumference of the circle, and x', y', z' the perpendiculars upon X, Y, Z respectively from the centre.

2. Let p be any point and X, Y, Z any straight lines in the same plane; let x, y, z denote respectively the perpendiculars from p upon X, Y, Z; and

let x', y', z' denote respectively the perpendiculars upon X, Y, Z from the opposite intersections YZ, ZX, XY. Then, whatever be the position of the point of reference P, we shall have

$$x \sin \mathrm{yz} + y \sin \mathrm{zx} + z \sin \mathrm{xy} = x' \sin \mathrm{yz} = y' \sin \mathrm{zx} = z' \sin \mathrm{xy},$$

and therefore $\quad -z \sin \mathrm{xy} = x \sin \mathrm{yz} + y \sin \mathrm{zx} - z' \sin \mathrm{xy}.$

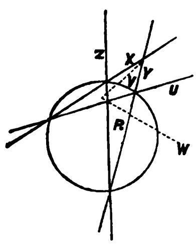
3. If the right-hand side of the last equation vanishes, the left-hand side must also vanish. This can only happen on the hypothesis that $z = 0$ or that $\sin \mathrm{xy} = 0$. The first hypothesis evidently restricts the position of p to the line Z; and the second, since it leaves z indeterminate, places no restriction whatever on the position of p. This last fact I overlooked, when I gave $(z) = x \pm y$ as a *necessary* consequence of the hypothesis $\sin \mathrm{xy} = 0$.

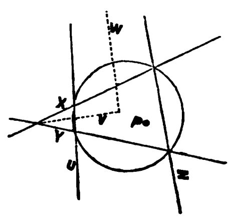
4. When the line Z bisects the angle xy we shall have $(z) = ax - ay$, in which a is any constant.

5. Let $(v) = ax + by + c$ and $(w) = a_1x + b_1y + c_1$ be the equations to V and W respectively. The straight lines V and W are parallel when $ab_1 = a_1b$, and at right angles when $aa_1 = -bb_1$.

We now proceed to the demonstration of Professor SYLVESTER's theorem, which may be otherwise enunciated as follows :—

If X, Y, Z, U be any straight lines whose intersections xz, zy, yu, ux are on the circumference of the same circle; then the bisectors, V and W say, of xy and zu are parallel or at right angles according to the position of the point of reference. The latter case will result from the position of P (the point of reference) in the subjoined figures.





Let p be any point in the plane; x, y, z, u its perpendicular distances from X, Y, Z, U respectively; and z', u' the perpendicular distances of the intersection xy from Z and U respectively. Then from (2) we get

$$-z \sin \mathrm{xy} = x \sin \mathrm{yz} + y \sin \mathrm{zx} - z' \sin \mathrm{xy} \quad \ldots\ldots\ldots\ldots \text{(A)};$$

and $$-u \sin \mathrm{xy} = x \sin \mathrm{yu} + y \sin \mathrm{ux} - u' \sin \mathrm{xy} \quad \ldots\ldots\ldots\ldots \text{(B)}.$$

But from the known relations between angles in the same segment of a circle and angles in opposite segments, we have $\sin \mathrm{yu} = -\sin \mathrm{zx}$, and $\sin \mathrm{ux} = -\sin \mathrm{yz}$. Substituting, therefore, in (B) and subtracting (B) from (A), we get

$$u \sin \mathrm{xy} - z \sin \mathrm{xy} = x\,(\sin \mathrm{yz} + \sin \mathrm{zx}) + y\,(\sin \mathrm{yz} + \sin \mathrm{zx})$$
$$+ (u' - z') \sin \mathrm{xy}.$$

If for convenience we denote the coefficient of u and z by m, that of x and y by n, and the other constant $(u'-z') \sin \mathrm{xy}$ by c, the last equation becomes

$$mu - mz = nx + ny + c.$$

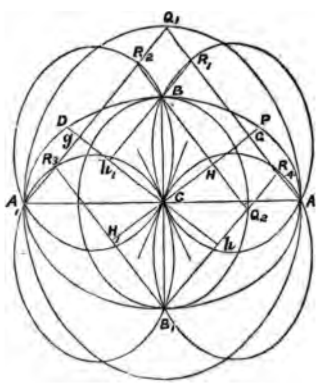
So far, no restriction has been put on the position of p; if now we restrict p

to the line W, the coordinates u and z become equal, and $mu-mz$ vanishes, so we get $(w) = mu - mz = nx + ny + c$.

Similarly if p be restricted to the line V, we get $(v) = ax - ay$; and since the coefficients of x and y in (v) and (w) satisfy the criterion in (5) the lines V and W are at right angles.

1922. (Proposed by W. GODWARD.)—Let AA_1, BB_1 be the major and minor axes of an ellipse, and CP, CD any pair of semi-conjugate diameters; draw AG, BH, B_1H_1 perpendicular to CP, and A_1g, B_1h, Bh_1 perpendicular to CD; also let AG, A_1g meet in Q_1; BH, B_1h in Q_2; AG, Bh_1 in R_1; A_1g, BH in R_2; A_1g, B_1H_1 in R_3; and AG, B_1h in R_4. Prove that the sum of the areas of the loci of Q_1, Q_2 is equal to the sum of the areas of the loci of R_1, R_2, R_3, R_4.

Solution by the PROPOSER; S. W. BROMFIELD; *the* REV. J. L. KITCHIN, M.A.; *and others.*



Let m be the tangent of the $\angle$ PCA, then, by conics, $\tan DCA = -\dfrac{b^2}{a^2m}$.

We have also the coordinates of

A...$(a, 0)$; A_1...$(-a, 0)$;

B...$(0, b)$; B_1...$(0, -b)$.

From which we at once obtain the equations to

$$AG \ldots y = -\frac{1}{m}(x-a) \quad \ldots\ldots\ldots(1);$$

$$A_1g \ldots y = \frac{a^2m}{b^2}(x+a) \quad \ldots\ldots\ldots(2);$$

$$BH \ldots y - b = -\frac{1}{m}x \quad \ldots\ldots\ldots(3);$$

$$B_1h \ldots y + b = \frac{a^2m}{b^2}x \quad \ldots\ldots\ldots\ldots(4);$$

$$B_1H_1 \ldots y + b = -\frac{1}{m}x \quad \ldots\ldots\ldots(5); \qquad Bh_1 \ldots y - b = \frac{a^2m}{b^2}x \quad \ldots\ldots\ldots\ldots(6).$$

Eliminating m, by direct multiplication, from (1) and (2), (3) and (4), (1) and (6), (2) and (3), (2) and (5), and (1) and (4) respectively, we have the equations of the loci of the following points, viz.,

$Q_1 \ldots a^2x^2 + b^2y^2 = a^4 \ldots\ldots(7)$; $\quad Q_2 \ldots a^2x^2 + b^2y^2 = b^4 \ldots\ldots(8)$

$R_1 \ldots a^2x^2 + b^2y^2 - a^3x - b^3y = 0 \ldots(9)$; $\quad R_2 \ldots a^2x^2 + b^2y^2 + a^3x - b^3y = 0 \; (10)$

$R_3 \ldots a^2x^2 + b^2y^2 + a^3x + b^3y = 0 \ldots(11)$; $\quad R_4 \ldots a^2x^2 + b^2y^2 - a^3x + b^3y = 0 \; (12)$.

We learn from (7) that the locus of Q_1 is an ellipse concentric with the primitive one, that its semi-axes are $\dfrac{a^2}{b}$ and a, and the area $= \dfrac{a^3}{b}\pi$, the major axis being perpendicular to AA_1. From (8) that the locus of Q_2 is also a concentric ellipse whose semi-axes are b and $\dfrac{b^2}{a}$ and area $= \dfrac{b^3}{a}\pi$, the major

axis being perpendicular to AA_1. And from (9), (10), (11), and (12) that the loci of R_1, R_2, R_3 and R_4 are also ellipses passing through (A, C, B), (A_1, C, B), (A_1, C, B_1), and (A, C, B_1) respectively, that their major axes are all perpendicular to AA_1, and that the several coordinates of their centres are $\left(\frac{a}{2}, \frac{b}{2}\right)$, $\left(-\frac{a}{2}, \frac{b}{2}\right)$, $\left(-\frac{a}{2}, -\frac{b}{2}\right)$, and $\left(\frac{a}{2}, -\frac{b}{2}\right)$; also that the semi-axes of each are $\frac{\sqrt{(a^4+b^4)}}{2b}$ and $\frac{\sqrt{(a^4+b^4)}}{2a}$, and area $= \frac{a^4+b^4}{4ab}\pi$.

It hence follows that the sum of the areas of the loci of Q_1 and Q_2 is $= \frac{a^4+b^4}{ab}\pi$ = the sum of the areas of the loci of R_1, R_2, R_3 and R_4.

Cor. 1.—Subtracting (9) from (11), we have $y = -\frac{a^3}{b^3}x$, the equation of the common tangent of the loci of R_1 and R_3. Also subtracting (10) from (12), we have $y = \frac{a^3}{b^3}x$ the equation to the common tangent of R_2 and R_4. It hence appears that the loci of R_1 and R_3, and likewise the loci of R_2 and R_4, have simple contact at C.

Cor. 2.—The rectangle of the major or of the minor axes of the ellipses which are the loci of Q_1 and Q_2, is equal to the square on the major or on the minor axis of the primitive ellipse.

Correction of an inaccuracy in Dr. Ingleby's Note on the Four-point Problem.

In my Note on this problem (*Reprint*, Vol. V., p. 81), I committed an inaccuracy which I now ask leave to correct. I wrote, "This problem has been variously solved by Professors Cayley, Sylvester, and Price." There seem to have been five distinct solutions. (1) That of Professor Sylvester: this was founded on a private communication from Professor Cayley, and was published by the former without the authority of the latter. This solution gave $\frac{1}{4}$. Professor Sylvester has seen reason to withdraw his acquiescence in this result, as virtually implying that infinite extent is bounded by a convex line, and Professor Cayley is probably in the same position. (2) That of Professor De Morgan, which gave $\frac{1}{2}$. (3) That founded on a principle employed by Mr. Woolhouse in the analogous question of three points in a plane forming an acute-angled triangle. This solution gives $\frac{35}{12\pi^2}$. (4) Mr. Wilson's solution, which gives $\frac{1}{3}$. (5) Another which gives $\frac{3}{8}$.

The problem arose in a sort of theory of points subordinate to Professor Sylvester's method of Compound Partitions, and was originally propounded by him in one of his lectures on Partitions, delivered at King's College. Professor Sylvester has been supposed to intimate his belief that the value of the probability in question (whether or not within assignable limits) is essentially indeterminate.

Mr. Wilson is right in saying of my Note that it seems to show that the probability is less than $\frac{1}{2}$, by a method which is incompetent to determine

how much less. My very object was to show that the value is $\frac{1}{8}$ minus a positive quantity that is *indeterminate*.

NOTE ON DR. INGLEBY'S STRICTURES ON MR. WILSON'S SOLUTION OF A PROBLEM IN CHANCES.

By the REV. PROFESSOR WHITWORTH.

Assuming Dr. Ingleby's figure and notation, we agree with him that the required chance in any configuration of the points is expressed by $\frac{1}{2}-\frac{\epsilon+\phi-\delta}{2(\alpha+\beta+\gamma)}$, where $\epsilon+\phi-\delta$ is a variable quantity dependent upon the configuration of the first three points.

The required probability will therefore be $\frac{1}{2}-x$, where x is a mean value of the variable fraction $\frac{\epsilon+\phi-\delta}{2(\alpha+\beta+\gamma)}$. In order to show that Mr. Wilson's solution is wrong, it would be necessary to show that the proper mean x cannot be $\frac{1}{8}$. Dr. Ingleby, however, only shows that the variable fraction may have values less than any assignable value, a fact by no means inconsistent with the supposition that its mean value is $\frac{1}{8}$.

In fact we agree with Mr. Wilson that Dr. Ingleby seems to have proved that the probability is less than $\frac{1}{2}$ by a method which is incapable of assigning how much less.

A SOLUTION OF THE PROBLEM OF DETERMINING THE PROBABILITY THAT FOUR POINTS TAKEN AT RANDOM IN A PLANE SHALL FORM A RE-ENTRANT QUADRILATERAL.

BY G. C. DE MORGAN, M.A.

If four points be taken at random upon a hyperbola, the probability that they shall form a re-entrant quadrilateral is $\frac{1}{2}$. For this will be the case if one point fall upon one branch and the other three on the other, and not otherwise; and there are 8 ways in which this may happen, and 16 is the whole number of ways of distributing the points between the two branches. Now the hyperbola whose equation is $x^2+axy-y^2+bx+cy+d=0$ may, by taking the coefficients properly, be made to pass through any set of four points; but we cannot in general pass two such hyperbolas through the same set. Hence we may consider all the different possible sets of four points as being made up of collections lying on the different hyperbolas represented by the equation above.

Since, whatever hyperbola the four fall on, the chance of a re-entrant quadrilateral is $\frac{1}{2}$, the chance on the supposition that they are taken anywhere in the plane, is $\frac{1}{2}$.

The cases where the four points are the intersections of two of the hyperbolas are not here taken into account, the chance that they shall be so being infinitely small. For, wherever three of them may fall, all such hyperbolas (in other words, all rectangular hyperbolas) passing through them intersect again in the same point, and the chance that the fourth point shall fall upon this intersection is infinitely small.

The same result is obtained by supposing each set of four points to consist of two pairs, separating all the possible pairs of points into collections lying on all possible different straight lines, and considering these straight lines two and two together.

NOTE ON QUESTION 1837. BY J. GRIFFITHS, M.A.

If we apply this theorem to any obtuse-angled triangle ABC and its self-conjugate circle, we have the following result. Let x be any point on one of the sides of the triangle (BC for instance), and let x be joined to the opposite vertex A; then the circle drawn on Ax as diameter is cut orthogonally by the self-conjugate circle. I may also remark that the polars of the feet of the perpendiculars of the triangle, with respect to the above circle, coincide with the lines drawn through the vertices A, B, C parallel to the opposite sides; whence we can easily construct other series of circles cut orthogonally by the self-conjugate circle.

1915. (Proposed by W. H. LAVERTY.)—

$$\left.\begin{aligned}\text{If}\quad C_r &= \frac{1}{x}\cos\frac{r\pi}{2m} + \frac{1}{x^2}\cos\frac{2r\pi}{2m} + \frac{1}{x^3}\cos\frac{3r\pi}{2m} + \&c.\\ \text{and}\quad S_r &= \frac{1}{x}\sin\frac{r\pi}{2m} + \frac{1}{x^2}\sin\frac{2r\pi}{2m} + \frac{1}{x^3}\sin\frac{3r\pi}{2m} + \&c.\end{aligned}\right\}\text{ to }\infty,\ \text{show that}$$

$$(P_1) = (C_1^2+S_1^2)(C_3^2+S_3^2)\ldots\ldots(C_{2m-1}^2+S_{2m-1}^2) = (x^{2m}+1)^{-1},$$

$$(P_2) = (C_0^2+S_0^2)^{\frac{1}{2}}(C_{2m}^2+S_{2m}^2)^{\frac{1}{2}}(C_2^2+S_2^2)\ldots(C_{2m-2}^2+S_{2m-2}^2) = (x^{2m}-1)^{-1}$$

Solution by S. W. BROMFIELD; REV. J. L. KITCHIN, M.A.; *the* PROPOSER; *and many others.*

Putting θ for $\frac{r\pi}{2m}$, and i as usual for $\sqrt{(-1)}$, we have

$$C_r^2+S_r^2 = (C_r+iS_r)(C_r-iS_r) = \left(\frac{e^{i\theta}}{x}+\frac{e^{2i\theta}}{x^2}+\ldots\right)\left(\frac{e^{-i\theta}}{x}+\ldots\right)$$

$$= \frac{e^{i\theta}}{x-e^{i\theta}}\cdot\frac{e^{-i\theta}}{x-e^{-i\theta}} = \frac{1}{x^2-x(e^{i\theta}+e^{-i\theta})+1} = \frac{1}{x^2-2x\cos\theta+1};$$

$$\therefore\quad (P_1)^{-1} = \left(x^2-2x\cos\frac{\pi}{2m}+1\right)\left(x^2-2x\cos\frac{3\pi}{2m}+1\right)\ldots\ldots = x^{2m}+1;$$

$$(P_2)^{-1} = (x^2-1)\left(x^2-2x\cos\frac{2\pi}{2m}+1\right)\left(x^2-2x\cos\frac{4\pi}{2m}+1\right)\ldots = x^{2m}-1.$$

1894. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.)—Supposing n chords to be drawn at random in a given circle, determine the several probabilities that there shall be 0, 1, 2, 3,..... $\frac{1}{2}n(n-1)$ intersections.

Solution by the PROPOSER.

Conceive the circumference of the circle to comprise 2N points equally distributed, the number 2N being indefinite. Then since a system of

chords formed by uniting one of these points with all the others in succession are distributed, in order, at equal angles round the common point, it is evident that a chord inflected at random must have an equal probability of passing through any one of the remaining $2N-1$ points. Therefore as regards the position of the chords in the circle the probabilities are precisely the same whether the chords are drawn at random or supposed to connect pairs of points arbitrarily taken.

Let F_n denote the total number of possible diagrams or configurations of n lines having their extremities chosen from amongst the $2N$ points. To determine the number F_n suppose a set of F_{n-1} diagrams to be drawn, each of them comprising $n-1$ lines and occupying $2n-2$ points. Then to convert any one of these into a diagram having n lines, the $2N-2n+2$ unoccupied points will admit of the formation of $\frac{1}{2}(2N-2n+2)(2N-2n+1) = (N-n+1)(2N-2n+1)$ independent lines, each of which will alike serve for the completion of such diagram. But after following out all these constructions it is evident that each particular diagram must ultimately be reproduced under the separate condition of each of its n lines appearing as the supplementary line, and must thereby become identically repeated n times. Hence by multiplying by the stated number of disposable lines and dividing by n for the purpose of excluding these repetitions, we find

$$F_n = \frac{N-n+1}{n}(2N-2n+1) \,.\, F_{n-1}$$

$$F_{n-1} = \frac{N-n+2}{n-1}(2N-2n+3) \,.\, F_{n-2}$$

$$\&c. \qquad\qquad \&c.$$

$$F_1 = \frac{N}{1}(2N-1)$$

$$\therefore\ F_n = \frac{N(N-1)\ldots(N-n+1)}{1\,.\,2\ldots n}\left\{(2N-1)(2N-3)\ldots(2N-2n+1)\right\} \quad \ldots(p).$$

When N is made equal to n, this becomes

$$f_n = 1\,.\,3\,.\,5 \ldots 2n-1 \ldots\ldots(m),$$

which is the total number of configurations of n lines occupying $2n$ points.

Again, by multiplying the value of f_n by the number of combinations $2n$ out of $2N$ points, we otherwise derive

$$F_n = \frac{2N(2N-1)\ldots(2N-2n+1)}{1\,.\,2\ldots 2n}(1\,.\,3\,.\,5\ldots 2n-1)\ldots\ldots(q).$$

This result is equivalent to the former; for

$$\frac{N(N-1)\ldots(N-n+1)}{1\,.\,2\ldots n}\left\{(2N-1)(2N-3)\ldots(2N-2n+1)\right\}$$

$$= \frac{\underline{|N}}{\underline{|n}\,\underline{|N-n}} \cdot \frac{1\,.\,3\,.\,5\ldots 2N-1}{1\,.\,3\,.\,5\ldots 2N-2n-1} = \frac{\underline{|2N}}{2^n\,\underline{|n}\,\underline{|2N-2n}}$$

$$= \frac{\underline{|2N}}{\underline{|2n}\,\underline{|2N-2n}} \cdot \frac{\underline{|2n}}{2^n\,\underline{|n}} = \frac{\underline{|2N}}{\underline{|2n}\,\underline{|2N-2n}} \cdot f_n.$$

It will be perceived that in the determination of the latter formula (q) the total number of configurations of the n chords is arrived at by first taking every combination $2n$ out of the $2N$ points; and in the next place, supposing each of these distinct systems of $2n$ points to be separately connected in pairs

in every possible way. Now, as the points are in every case all posited in the periphery of a closed curve, the exterior of which is everywhere convex, a little consideration will show that, as regards intersections and varieties of configuration, each individual system of $2n$ points will yield precisely the same results. It will hence be sufficient to consider only one set of $2n$ points, and the particular position of these points will evidently be quite immaterial. Therefore, generally, whatever be the number 2N, whether finite or infinite, the probabilities required in the question are identical with those which appertain to conditions which are thus simplified and reduced down to the following:—

Let there be given $2n$ points which if joined in consecutive order would form any convex polygon whatever of as many sides; then supposing pairs of these points to be arbitrarily taken and the same to be severally united in n lines, determine the respective probabilities that amongst these n lines there shall be $0, 1, 2, 3, \ldots \frac{1}{2}n(n-1)$ intersections.

Here the problem is entirely divested of considerations which involve the integral calculus, and now appertains to ordinary algebra alone, since the total number of ways is reduced to a finite and determinate number, being in fact the number of ways in which the $2n$ points can be associated in pairs, and this we have already found is expressed by the factorial $1 \,.\, 3 \,.\, 5 \ldots 2n-1$.

When the number n is small, the various combinations may be readily put down, either geometrically or symbolically, and the numbers of intersections respectively counted. To abbreviate and simplify this last operation, let the several points, taken in consecutive order, be denoted by the numerals $1, 2, 3, \ldots. 2n$; and in denoting a line let the smaller numeral be always placed first, and let the set of lines which compose each combination or diagram be so arranged that these leading numerals shall proceed in the order of magnitude. When a combination of lines is put down in this convenient manner, the fact of the intersection or non-intersection of any stated line with each of those that follow it will be made apparent by simply noting whether the magnitude of the terminal numeral of the same be included or not between the two numerals which indicate each subsequent line.

As a first example, take the simplest case, viz., that of two chords, which was originally proposed by me as Quest. 1904, in the *Lady's and Gentleman's Diary* for 1856, and solved by Dr. Rutherford and Mr. Miller in the *Diary* for 1857 (pp. 55, 56). According to what precedes, we shall have $2n = 4$ points, and $1 \,.\, 3 = 3$ combinations or diagrams, which may be stated thus:—

12, 34	having	0	intersection
13, 24	„	1	„
14, 23	„	0	„

As these exhibit two diagrams without intersection and one diagram with intersection, the probabilities of intersection and non-intersection of two arbitrary chords are $\frac{1}{3}$ and $\frac{2}{3}$.

As another example, take the case of three chords, Quest. 1152 of the *Educational Times*, to which an effective solution by the integral calculus is given by Mr. Miller, the Editor, in the *Reprint*, Vol. II., p. 92. Here we have $2n = 6$ points, and $1 \,.\, 3 \,.\, 5 = 15$ combinations, viz.,

12, 34, 56	having	0	intersections
35, 46	„	1	„
36, 45	„	0	„
13, 24, 56	„	1	„
25, 46	„	2	„
26, 45	„	1	„

14, 23, 56	having	0	intersections
25, 36	"	3	"
26, 35	"	2	"
15, 23, 46	"	1	"
24, 36	"	2	"
26, 34	"	1	"
16, 23, 45	"	0	"
24, 35	"	1	"
25, 34	"	0	"

This complete system of combinations consists of the following summary

5	diagrams having	0	intersections
6	"	1	"
3	"	2	"
1	"	3	"

Therefore in this case the respective probabilities of 0, 1, 2, 3 intersections are $\frac{1}{3}$, $\frac{2}{5}$, $\frac{1}{5}$, $\frac{1}{15}$.

It will be convenient to express the numbers by means of the coefficients of the several terms of an algebraic function, in which the exponent of x is made to denote the number of intersecting pairs.

The solutions to the two foregoing examples when thus stated are,—

$$I_2 = 2 + x$$
$$I_3 = 5 + 6x + 3x^2 + x^3,$$

in which the coefficient of x^r represents the number of diagrams which have r intersections.

The general solution of the proposed problem will hence be included in a method of determining the function I_n in the general case.

By a somewhat elaborate process I have arrived at the following remarkable results.

Let $1, \delta_1, \delta_2, \delta_3, \ldots . \delta_n$ be the first differences of the coefficients of the binomial $(1+x)^{2n}$ taken as far as the central or maximum coefficient; also let

$$\nu = \frac{(n+1)\,n}{2}, \quad \nu' = \frac{n\,(n-1)}{2}, \quad \nu'' = \frac{(n-1)\,(n-2)}{2}. \ \&\text{c.}$$

$$\left.\begin{aligned} \text{Then } I_n &= \frac{\delta_n - \delta_{n-1}\,x + \delta_{n-2}\,x^3 - \delta_{n-3}\,x^6 + \delta_{n-4}\,x^{10} - \&\text{c.}}{(1-x)^n} \\ &= \frac{x^{\nu} - \delta_1 x^{\nu'} + \delta_2 x^{\nu''} - \delta_3 x^{\nu'''} + \&\text{c.}}{(x-1)^n} \end{aligned}\right\} \ldots\ldots (a)$$

the result of each division being always an exact function, or one which leaves no remainder. Perhaps the most expeditious form for numerical calculation is

$$I_n = \left(1 - \frac{1}{x}\right)^{-n} \left(x^{\nu-n} - \delta_1 x^{\nu'-n} + \delta_2 x^{\nu''-n} - \delta_3 x^{\nu'''-n} + \&\text{c.}\right)$$
$$= \left(1 + nx^{-1} + n\frac{n+1}{2}x^{-2} + n\,\frac{n+1}{2}\,\frac{n+2}{3}x^{-3} + \&\text{c.}\right)$$
$$\times \left(x^{\nu-n} - \delta_1 x^{\nu'-n} + \delta_2 x^{\nu''-n} - \delta_3 x^{\nu'''-n} + \&\text{c.}\right) \ldots\ldots\ldots (b)$$

where the coefficients of the first factor, being the ordinary figurate numbers, are easily formed into a preliminary table by successive addition. Also the terms of the second factor may be rejected as soon as the exponent becomes

negative. since the coefficients of negative exponents must necessarily become neutralized in the final result, which will be of the form

$$\mathrm{I}_n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots\ldots + c_s x^s,$$

where $s = \dfrac{n(n-1)}{2}$.

The several coefficients of $(1-x)^n . \mathrm{I}_n$ must obviously be the expanded nth differences of the coefficients of I_n, and therefore by (a) these latter must consist of the series of values

$$\delta_n | -\delta_{n-1} | 0 | + \delta_{n-2} | 0, 0 | - \delta_{n-3} | 0, 0, 0 | + \delta_{n-4}, \text{\&c.} \;\ldots\ldots (\alpha).$$

The coefficients of I_n may therefore be numerically determined from this series of differences by making n successive summations, which are, in fact, equivalent to n successive divisions by $1-x$. Or the first step of this process may be dispensed with as follows:—Let $\alpha_1, \alpha_2, \alpha_3, \ldots . \alpha_n$ be the first differences of the first n coefficients of $(1+x)^{2n-1}$; then will the $(n-1)$th order of differences of the coefficients of I_n be

$$\alpha_n | -\alpha_{n-1} - \alpha_{n-1} | + \alpha_{n-2} + \alpha_{n-2} + \alpha_{n-2} | - \text{\&c. \&c.} \ldots\ldots\ldots (\beta),$$

in which series the signs alternate with each group and the repetitions are successively increased by unity. From these differences the coefficients of I_n are hence determined by $n-1$ successive summations, and the whole operation will occupy just n columns.

As an example of the formula (a) let $n = 5$; then

$$(1+x)^{10} = 1 + 10x + 45x^2 + 120x^3 + 210x^4 + 252x^5 + \text{\&c.},$$

$$\therefore \text{ values of } \delta_{0 \ldots n} = 1, 9, 35, 75, 90, 42;$$

$$\begin{aligned} \text{and } \mathrm{I}_5 &= \frac{42 - 90x + 75x^3 - 35x^6 + 9x^{10} - x^{15}}{(1-x)^5} \\ &= 42 + 120x + 180x^2 + 195x^3 + 165x^4 + 117x^5 \\ &\quad + 70x^6 + 35x^7 + 15x^8 + 5x^9 + x^{10}. \end{aligned}$$

Or, by the formula (b)

$$\mathrm{I}_5 = \left(1 + \frac{5}{x} + \frac{15}{x^2} + \frac{35}{x^3} + \frac{70}{x^4} + \frac{126}{x^5} + \frac{210}{x^6} + \frac{330}{x^7} + \frac{495}{x^8} + \frac{715}{x^9} + \frac{1001}{x^{10}} + \text{\&c.}\right) \times (x^{10} - 9x^5 + 35x - \text{\&c.})$$

$$\begin{array}{r} = x^{10} + 5x^9 + 15x^8 + 35x^7 + 70x^6 + 126x^5 + 210x^4 + 330x^3 + 495x^2 + 715x + 1001 \\ -9x^5 - 45x^4 - 135x^3 - 315x^2 - 630x - 1134 \\ +\ 35x +\ 175 \\ \hline = x^{10} + 5x^9 + 15x^8 + 35x^7 + 70x^6 + 117x^5 + 165x^4 + 195x^3 + 180x^2 + 120x + \ 42 \end{array}$$

And this is undoubtedly the most concise method of obtaining the function I_n when the series of figurate numbers which appear in the first factor are previously tabulated.

To apply the formula (β), we have

$$(1+x)^9 = 1 + 9x + 36x^2 + 84x^3 + 126x^4 + 126x^5 + \text{\&c.};$$

$$\text{therefore values of } \alpha_{1 \ldots n} = 1, 8, 27, 48, 42.$$

Hence by placing the series of differences in vertical columns the successive summations will appear as follows, and the accuracy of the results will be made apparent by their regular subsidence at the end of each column, the same being, in each case, indicated by an asterisk (*) :

a				Diagrams c.	Intersections
42	42	42	42	42	0
−48	−6	36	78	120	1
−48	54	−18	60	180	2
+27	−27	45	15	195	3
27	0	45	−30	165	4
+27	+27	−18	48	117	5
−8	19	+1	47	70	6
8	11	12	35	35	7
8	+3	15	20	15	8
−8	−5	10	10	5	9
+1	−4	+6	−4	1	10
1	3	3	−1	*	
1	2	+1	*		
1	−1	*			
+1	*				

The numbers in each column are successively deduced by the algebraic addition of the corresponding number in the preceding column ; and those which extend below the horizontal line may evidently be omitted, in which case the general accuracy of the work will be satisfactorily tested by observing that the last horizontal line should then comprise the coefficients of $(1+x)^{n-1}$ with alternate signs.

If the function I_n be put down with its terms in a reverse order, or as immediately derived from the second of the formulæ (a), thus :

$$I_n = c_s x^s + c_{s-1} x^{s-1} + c_{s-2} x^{s-2} + \&c. ;$$

then the coefficients c_s, c_{s-1}, c_{s-2}, &c., being in a reverse order, their $(n-1)$th order of differences will be the reverse of the foregoing, viz.,

$+1$ (n times) ; $-a_1$ ($n-1$ times) ; $+a_2$ ($n-2$ times) ; &c. &c.

and hence the coefficients may be determined as before.

Thus, returning to the same example :—

a				Diagrams c.	Intersections
+1	+1	+1	+1	1	10
1	2	3	4	5	9
1	3	6	10	15	8
1	4	10	20	35	7
+1	+5	15	35	70	6
−8	−3	12	47	117	5
8	11	+1	48	165	4
8	19	−18	+30	195	3
−8	−27	45	−15	180	2
+27	0	45	60	120	1
27	+27	−18	78	42	0
+27	54	+36	−42	*	
−48	+6	42	*		
−48	−42	*			
+42	*				

In all these summations it is curious to observe the persistence of negative signs until the final column is attained.

To adduce other examples, and to give a facility of practical reference, the results up to as many as eight chords are shown in the following table:—

TABULATED NUMBER OF CONFIGURATIONS (c).

Number of Chords (n)							Inter-sections
2	3	4	5	6	7	8	
2	5	14	42	132	429	1430	0
1	6	28	120	495	2002	8008	1
	3	28	180	990	5005	24024	2
	1	20	195	1430	9009	51688	3
		10	165	1650	13013	89180	4
		4	117	1617	16016	131040	5
		1	70	1386	17381	169988	6
			35	1056	16991	199264	7
			15	726	15197	214578	8
			5	451	12558	214760	9
			1	252	9646	201460	10
				126	6916	178248	11
				56	4641	149464	12
				21	2912	119168	13
				6	1703	90540	14
				1	924	65640	15
					462	45438	16
					210	30024	17
					84	18908	18
					28	11320	19
					7	6420	20
					1	3432	21
						1716	22
						792	23
						330	24
						120	25
						36	26
						8	27
						1	28
3	15	105	945	10395	135135	2027025	Totals

To find the probability of any proposed number of intersections, the number taken from this table is to be divided by the total given at the foot of the column.

The mathematical relations amongst the coefficients (c) may, according to what has preceded, be expressed under the form of an nth difference. Let r

be any number from 0 to $n\frac{n-1}{2}$; and

$$c_r - nc_{r-1} + n\frac{n-1}{2}c_{r-2} - n\frac{n-1}{2}\frac{n-2}{3}c_{r-3} + \&c. = \mathrm{R}.$$

Then when r is of the form $k\frac{k+1}{2}$, k being any integral number $< n$, we shall have

$$\mathrm{R} = (-1)^k \delta_{n-k} = (-)^k\left(\frac{2n\ldots n+k+1}{1\,.\,.\,n-k} - \frac{2n\ldots n+k+2}{1\,.\,.\,n-k-1}\right)$$

$$= (-)^k \frac{2k+1}{n+k+1}\cdot\frac{\underline{|2n}}{\underline{|n-k}\;\underline{|n+k}}.$$

And when r is not of the form $k\frac{k+1}{2}$; then $\mathrm{R} = 0$.

Also $c_r = 0$ for all values of $r < 0$ or $> n\frac{n-1}{2}$.

Take $r = 0$; it is of the form $k\frac{k+1}{2}$, and $k = 0$,

therefore $$c_0 = \mathrm{R} = \frac{\underline{|2n}}{\underline{|n}\;\underline{|n+1}},$$

and hence the probability of the general case of non-intersection is

$$p_0 = \frac{c_0}{1.3.5\,.\,.\,2n-1} = \frac{2^n}{\underline{|n+1}}.$$

When $r = 1$ it is again of the form $k\frac{k+1}{2}$, and $k = 1$;

therefore $$\mathrm{R} = -\frac{3}{n+2}\cdot\frac{\underline{|2n}}{\underline{|n-1}\;\underline{|n+1}}, \text{ and } c_1 - nc_0 = \mathrm{R};$$

therefore $$c_1 = nc_0 + \mathrm{R} = \left(1 - \frac{3}{n+2}\right)\frac{\underline{|2n}}{\underline{|n-1}\;\underline{|n+1}}$$

$$= \frac{n-1}{n+2}\frac{\underline{|2n}}{\underline{|n-1}\;\underline{|n+1}} = \frac{\underline{|2n}}{\underline{|n-2}\;\underline{|n+2}};$$

which result exhibits the remarkable property, that the number of configurations having each of them but one intersection is equal to the number of combinations of $n-2$ or $n+2$ out of the $2n$ points.

When $r=2$ it is not of the form $k\frac{k+1}{2}$;

therefore $$\mathrm{R} = 0, \text{ and } c_2 = nc_1 + n\frac{n-1}{2}c_0 = 0,$$

giving $$c_2 = n\left(c_1 - \frac{n-1}{2}c_0\right) = (n-1)\left(\frac{n}{n+2} - \frac{1}{2}\right)\frac{\underline{|2n}}{\underline{|n-1}\;\underline{|n+1}}$$

$$= \frac{n-1)(n-2)}{2(n+2)} \cdot \frac{\underline{|2n}}{\underline{|n-1}\,\underline{|n+1}} = \frac{n-2}{2} \cdot \frac{\underline{|2n}}{\underline{|n-2}\,\underline{|n+2}}.$$

The general combinations for specific intersections may be otherwise deduced more directly from the formula (b). Thus, the coefficients of the binomial $(1+x)^{2n}$ are

$$\beta_n = \frac{n+1\ldots 2n}{1\ldots n}, \quad \beta_{n-1} = \frac{n+2\ldots 2n}{1\ldots n-1}, \quad \beta_{n-2} = \frac{n+3\ldots 2n}{1\ldots n-2},$$

$$\beta_{n-3} = \frac{n+4\ldots 2n}{1\ldots n-3}, \quad \beta_{n-4} = \frac{n+5\ldots 2n}{1\ldots n-4}, \text{ \&c.}$$

therefore $\delta_n = \beta_n \;\; -\beta_{n-1} = \dfrac{n+2\ldots 2n}{1\ldots n}$

$$\delta_{n-1} = \beta_{n-1} - \beta_{n-2} = \frac{3(n+3\ldots 2n)}{1\ldots n-1} = \frac{3n}{n+2}\,\delta_n$$

$$\delta_{n-2} = \beta_{n-2} - \beta_{n-3} = \frac{5(n+4\ldots 2n)}{1\ldots n-2} = \frac{5n\,(n-1)}{(n+2)\,(n+3)}\,\delta_n$$

$$\delta_{n-3} = \beta_{n-3} - \beta_{n-4} = \frac{7(n+5\ldots 2n)}{1\ldots n-3} = \frac{7n\,(n-1)\,(n-2)}{(n+2)\,(n+3)\,(n+4)}\,\delta_n$$

&c. &c. &c.

Hence, according to (b), we find

$$c_0 = \delta_n = \frac{n+2\ldots 2n}{1\ldots n}$$

$$c_1 = n\delta_n - \delta_{n-1} = \beta_{n-2} = \frac{n+3\ldots 2n}{1\ldots n-2}$$

$$c_2 = n\frac{n+1}{2}\,\delta_n - n\delta_{n-1} = \frac{n+3\ldots 2n}{2(1\ldots n-3)}$$

$$c_3 = n\frac{n+1}{2}\cdot\frac{n+2}{3}\,\delta_n - n\,\frac{n+1}{2}\,\delta_{n-1} + \delta_{n-2}$$

$$= \frac{n^2+2n-9}{2.3}\cdot\frac{n+4\ldots 2n}{1\ldots n-3}$$

$$c_4 = \frac{n\ldots n+3}{2.3.4}\,\delta_n - \frac{n\,(n+1)\,(n+2)}{2.3.4}\,\delta_{n-1} + n\delta_{n-2}$$

$$= \frac{(n-1)\,(n+6)}{2.3.4}\cdot\frac{n+4\ldots 2n}{1\ldots n-4}$$

$$c_5 = \frac{n\ldots n+4}{2\ldots 5}\,\delta_n - \frac{n\ldots n+3}{2.3.4}\,\delta_{n-1} + n\,\frac{n+1}{2}\,\delta_{n-2}$$

$$= \frac{(n+1)\,(n-3)\,(n+8)}{2.3.4.5}\cdot\frac{n+4\ldots 2n}{1\ldots n-4}$$

$$c_6 = \frac{n\ldots n+5}{2\ldots 6}\,\delta_n - \frac{n\ldots n+4}{2\ldots 5}\,\delta_{n-1} + \frac{n\,(n+1)\,(n+2)}{2.3}\,\delta_{n-2} - \delta_{n-3}$$

$$= (n-2)\,\frac{n^4+14n^3+29n^2-164n-600}{2.3.4.5.6}\cdot\frac{n+5\ldots 2n}{1\ldots n-4}$$

$$c_7 = \frac{n\ldots n+6}{2\ldots7}\delta_n - \frac{n\ldots n+5}{2\ldots6}\delta_{n-1} + \frac{n\ldots n+3}{2.3.4}\delta_{n-2} - n\delta_{n-3}$$

$$= \frac{n^5+19n^4+83n^3-97n^2-1206n+720}{2.3.4.5.6.7} \cdot \frac{n+5\ldots2n}{1\ldots n-5}$$

&c. &c.

$$c_r = \frac{n\ldots n+r-1}{1\ldots r}\delta_n - \frac{n\ldots n+r-2}{1\ldots r-1}\delta_{n-1} + \frac{n\ldots n+r-4}{1\ldots r-3}\delta_{n-2}$$

$$- \frac{n\ldots n+r-7}{1\ldots r-6}\delta_{n-3} + \text{\&c.}$$

where the coefficients of δ_n, δ_{n-1}, δ_{n-2}, &c., are figurate numbers which first descend one degree, then two degrees, then three degrees, and so on, by a uniformly widening interval of succession.

By dividing these values of c by

$$\Sigma c = 1.3.5\ldots2n-1 = \frac{1\ldots2n}{2^n(1\ldots n)} = \frac{n+1\ldots2n}{2^n}$$

the corresponding probabilities, for stated intersections, are finally determined to be

$$p_0 = \frac{2^n}{\underline{|n+1}}$$

$$p_1 = \frac{1}{(n+1)(n+2)} \cdot \frac{2^n}{\underline{|n-2}}$$

$$p_2 = \frac{1}{(n+1)(n+2)} \cdot \frac{2^{n-1}}{\underline{|n-3}}$$

$$p_3 = \frac{n^2+2n-9}{(n+1)(n+2)(n+3)} \cdot \frac{2^{n-1}}{3\underline{|n-3}}$$

$$p_4 = \frac{(n-1)(n+6)}{(n+1)(n+2)(n+3)} \cdot \frac{2^{n-3}}{3\underline{|n-4}}$$

$$p_5 = \frac{(n-3)(n+8)}{(n+2)(n+3)} \cdot \frac{2^{n-3}}{15\underline{|n-4}}$$

$$p_6 = \frac{n^4+14n^3+29n^2-164n-600}{(n+1)(n+2)(n+3)(n+4)} \cdot \frac{2^{n-4}(n-2)}{45\underline{|n-4}}$$

$$p_7 = \frac{n^5+19n^4+83n^3-97n^2-1206n+720}{(n+1)(n+2)(n+3)(n+4)} \cdot \frac{2^{n-4}}{315\underline{|n-5}}$$

&c. &c.

$$p_r = \frac{(r+1\ldots r+n-1)\delta_n - (r\ldots r+n-2)\delta_{n-1} + (r-2\ldots r+n-4)\delta_{n-2} - \text{\&c.}}{(1\ldots n-1)(1.3\ldots2n-1)}$$

The general case of non-intersection may be otherwise calculated, for successive values of n, by another method in which the reasoning is of the most palpable and elementary kind. Suppose a set of diagrams to be formed from $2(n+1)$ points, each of them comprising $n+1$ non-intersecting lines, and let the number of these diagrams be denoted by ϕ_{n+1}. All the varieties of diagram, which make up the set, may obviously be attained, inclusively,

by the following considerations. First, conceive a certain partition of diagrams to be chosen such that all of them shall contain one particular line, and refer these several diagrams to this common line, which may be designated the $(n+1)$th line. Then, as there is no intersection, it will appear that the other n lines must be either wholly on one side, or distributed in a specific manner on both sides of this $(n+1)$th line. That is, if there be x lines on one side of it, there will be $n-x$ lines on the other side, and the number x, to include all partitions, may have the several values $0, 1, 2, 3, \ldots n$. Now, following out our notation, the x non-intersecting lines, which unite $2x$ points on one side, may be drawn in ϕ_x ways; and the $n-x$ lines on the other side may in like manner be drawn in ϕ_{n-x} ways. And as each of the former of these constructions may be united with each of the latter, the number of complete diagrams which thereby result, under the stated hypothesis of distribution, will be $\phi_x \phi_{n-x}$. Hence by giving to x all its possible values, as before stated, the total number of non-intersecting diagrams is found to be

$$\phi_{n+1} = \Sigma \phi_x \phi_{n-x}$$
$$= \phi_0 \phi_n + \phi_1 \phi_{n-1} + \phi_2 \phi_{n-2} + \ldots + \phi_n \phi_0$$

in which $\phi_0 = 1$.

By means of this general formula the value of the function for any proposed number is immediately deduced from the values previously determined for all inferior numbers. Thus we get, successively,

$$\begin{aligned}
\phi_1 &= 1.1 \ldots\ldots\ldots\ldots\ldots\ldots = 1 \\
\phi_2 &= 1.1 + 1.1 \ldots\ldots\ldots\ldots\ldots = 2 \\
\phi_3 &= 1.2 + 1.1 + 2.1 \ldots\ldots\ldots\ldots = 5 \\
\phi_4 &= 1.5 + 1.2 + 2.1 + 5.1 \ldots\ldots\ldots = 14 \\
\phi_5 &= 1.14 + 1.5 + 2.2 + 5.1 + 14.1 \ldots\ldots = 42 \\
\phi_6 &= 1.42 + 1.14 + 2.5 + 5.2 + 14\ 1 + 42.1 = 132 \\
& \&c. \qquad \&c. \qquad \&c.
\end{aligned}$$

To determine f_{n+1} the total number of diagrams, both intersecting and non-intersecting, we observe that, as the several lines may or may not intersect the $(n+1)$th line, it is no longer requisite to have a distinct set of lines, or an even number of points, on each side of it. The points may therefore be distributed in any manner on each side of the $(n+1)$th line. If there be z points on one side of it, there will be $2n-z$ points on the other side, and the number z may take any of the values $0, 1, 2, 3, \ldots 2n$. Also under each hypothesis of distribution the z and $2n-z$ points respectively on each side of the $(n+1)$th line make up a system of $2n$ points which may be joined in pairs by n lines in f_n ways, the same being independent of the value of z. Therefore, as z admits of having $2n+1$ values, the total number of diagrams of $n+1$ lines, occupying $2(n+1)$ points, under every arrangement, is

$$f_{n+1} = (2n+1) f_n$$

from which we readily infer that

$$f_{n+1} = 1.3.5 \cdots\cdots (2n+1)$$
$$f_n = 1.3.5 \ldots\ldots (2n-1)$$

the same result as before obtained.

The probability of non-intersection is of course found by dividing the value of ϕ by the corresponding value of f.

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1674. A cylinder filled with fluid is closed at both ends, and then suspended from a point in the rim of one end; find the resultant pressure on the curved surface, and prove that the direction of this resultant pressure, and the axis of the cylinder, make equal angles with the vertical and horizontal respectively. 78

1676. A hollow cone without weight, closed and filled with water, is suspended from a point in the rim of its base; prove that (1), if α be the semivertical angle of the cone, the total pressures on the curved surface and the base are in the ratio $(1+11\sin^2\alpha) : 12\sin^3\alpha$; and (2), if ϕ be the angle which the direction of the resultant pressure on the curved surface makes with the vertical, $\cot\phi = \frac{1}{48}(\cot^3\alpha + 28\cot\alpha)$ 79

1693. Let $\Pi(m, \omega)$, $\Pi(m, \phi)$, $\Pi(m, \psi)$ be three arcs of a parabola measured from the vertex; ω, ϕ, and ψ their amplitudes (that is, the inclinations to the axis of the focal perpendiculars on the tangents at the extremities of the arcs) being connected by the equation $\tan\omega = \tan\phi\sec\psi + \sec\phi\tan\psi$. Show that their algebraic sum is equal to the product of the ordinates at their extremities divided by the square of the semi-parameter; and apply the theorem to the particular case in which $\tan\omega = 2$, $\tan\phi = \frac{1}{2}$, $\tan\psi = \frac{1}{2}\sqrt{5}$ 62

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1729. Prove that the difference between any parabolic arc (measured from the vertex) and its ordinate is equal to the arc cut off by a focal chord, the amplitudes being connected by the condition $\tan\omega = \sec\phi + \operatorname{cosec}\phi$. 62

1736. Any number of chords of a given fixed plane curve are drawn through a fixed point, and are traversed by a number of perfectly elastic particles, moving from rest at the fixed point; find the form of the curve, if, after impact on the perfectly smooth arc, all the particles pass through another fixed point vertically below the former. 29

1759. Travelling at night on a line of railway of circular form, I noticed three distant lights L, L′, L″. At a certain station A, the lights L and L′ appeared to coalesce; and at the next station B, the lights L and L″ assumed the same appear-

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ance; the distances LL′, LL″, L′L″ are k, l, m, respectively, the direct distance from A to B is n, and the circle of which the Railway forms a part would if completed pass through L. Now if with radius of this circle and centre A, another circle were described, cutting the Railway between A and B in Q, find the distance along the line from Q to B.................... 48

1777. Prove that

$$\Delta^r \log^m (1+\Delta)\, 0^n = n(n-1)\ldots.(n-m+1)\,\Delta^r 0^{n-m}.$$

N.B.—From this formula, when r is positive and less than m, by putting $n+m$ for n, we get

$$(n+1)(n+2)\ldots(n+m)\,\Delta^{-r}0^n = \Delta^{m-r}(1-\tfrac{1}{2}\Delta+\tfrac{1}{3}\Delta^2-\&c.)^m 0^{n+m}.$$ 85

1787. On donne une conique K et un point p. Une transversale menée arbitrairement par p recontre K en deux points m, m'; et soit x un point de la transversale tel que le rapport anharmonique $(pxmm')$ soit un nombre λ donné. Trouver le lieu du point x. Si λ est l'une des racines cubiques imaginaires de -1, on a une certaine conique C (p). De quelle manière change C (p), si l'on fait varier p?

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1793. 1. If two sides of a triangle be given, and the third side be taken at random (from among all its possible values), find the probability that the triangle is acute-angled.

2. Two points are taken at random in a given line (l); find the probability of the distance between them exceeding a given length (c).

3. In a plane triangle if arbitrary values be taken for a, b, and C (two sides and the contained angle), and the extreme limit be the same for a and b; find the probability that the triangle is obtuse-angled. 97

1799. Three conics are described so that each of them passes through the same point O, and through the extremities of two of the diagonals of the same complete quadrilateral. Prove that if O_1, O_2, O_3 are their other points of intersection, then OO_1, OO_2, OO_3 are the tangents to the three conics at O. 54

1808. Décomposer un nombre triangulaire en d'autres nombres triangulaires, dans toutes les manières possibles. 18

1809. On circonscrit à un triangle quelconque une courbe du second degré telle que les normales aux trois sommets du triangle passent par un point. On demande de prouver que le lieu de ce point est une courbe à centre du troisième ordre. Déterminer cette courbe.. 109

1839. If normals be drawn to a conic at the points P, Q; show that a parabola can be described, touching these two normals, the chord PQ, and the axes of the conic; the diameter conjugate to the chord being the directrix. Also verify the following determination of the common tangents to these curves; through the pole of the chord draw the four normals to the conic, the tangents at their feet are the common tangents required... 108

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1844. If $(\epsilon)_1$ represent ϵ^x, $(\epsilon)_2$ represent ϵ^{ϵ^x}, &c.; and if $\log^2 x$ represent $\log_\epsilon(\log_\epsilon x)$, &c.; find the value of

$$\int_0^\infty \frac{(\epsilon)_{n-1}\cdot(\epsilon)_{n-2}\cdots(\epsilon)_1 \cdot dx}{(\epsilon)_n \cdot \sqrt{(\epsilon)_{n-1}}}\,;\ \int_0^\infty \frac{\log^n(x)\,.\,dx}{\left\{\log^{n-1}(x)\right\}^2 . \log^{n-2}(x)\ldots\log(x)\,.\,x}$$ 80

1853. Find two series of integral cubes such that every term in the first may be the sum, and every term in the second the difference, of two integral squares. Also find two series of integral squares, such that every term in the first may be the sum, and every term in the second the difference, of two integral cubes. 90

1862. Determine a system of values for (x, y, z), functions of (α, β, γ), and satisfying identically the equation $x^3+y^3+z^3-3xyz=(\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma)^2$. 24

1865. Find the equation and parameter of the parabola osculating most closely at the origin the conic

$$ax+by+cx^2+2dxy+ey^2=0\ldots\ldots\ldots(1);$$

and find also the angle (θ) between the axis of x and the axis of the required parabola. 77

1872. Show that the surfaces $xyz=1$, $yz+zx+xy+x+y+z+3=0$, intersect in two distinct cubic curves; and find the equations of the cubic cones which have their vertices at the origin and pass through these curves respectively. 65

1878. A line of length a is broken up into n pieces at random; prove that (1) the chance that they cannot be made in a polygon of n sides is $n2^{1-n}$; and (2) the chance that the sum of the squares described on them does not exceed $\frac{a^2}{n-1}$ is $\left(\frac{\pi}{n^2-n}\right)^{\frac{1}{2}(n-1)} \frac{\Gamma(n)}{\Gamma\left\{\frac{1}{2}(n+1)\right\}}\cdot\frac{1}{n^{\frac{1}{2}}}$. 83

1881. Let O be the middle point of any chord AA′ of a circle ABA′; through O draw any other chord BOB′; join AB, A′B′, and produce these lines to meet in P. Show that the locus of P is a straight line parallel to AA′; and is the same as that of the intersection of tangents drawn to the circle at B, B′. 44

1892. Observing that the form

$$\left\{(b^2-a^2)x+bcy+c^2z\right\}^2-2d^2\left\{(b^2+a^2)x+bcy+c^2z\right\}z+d^4z^2$$

is a function solely of $x, y, z, a^2-b^2, c^2-d^2, bc, ad$; show that if

$$\alpha^2=\frac{b^2-a^2}{c^2-d^2}d^2,\quad \beta^2=\frac{b^2-a^2}{c^2-d^2}c^2,\quad \gamma^2=\frac{c^2-d^2}{b^2-a^2}b^2,\quad \delta^2=\frac{c^2-d^2}{b^2-a^2}a^2,$$

and A, B, B′ be three points in a straight line such that $\mathrm{AB}=\frac{c}{b}$, $\mathrm{AB}'=\frac{\gamma}{\beta}$; then, if any point P be found satisfying the equation $a\,.\,\mathrm{AP}+b\,.\,\mathrm{BP}=d$, on giving right signs to α, β, the equation $\alpha\,.\,\mathrm{AP}+\beta\,.\,\mathrm{B'P}=\delta$ will also be satisfied. 66

1896. Show that the lines trisecting an angle of a triangle do not trisect the opposite side. 43

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1899. If perpendiculars from any point P on a hyperbola are drawn to the asymptotes, prove that the line joining the feet of perpendiculars from any other point Q on the curve to these lines passes through a fixed point F. Also find the locus of F, as P varies on the curve, and determine the hyperbola for which this locus reduces to a point. 45

1900. The Austrian Government have lately issued a loan of 734,694 Bonds of £19. 17*s*. 0*d*. sterling, or 500 francs, or 200 florins Austrian, value in silver; and it appears that a contract for it has been entered into between the Imperial Government of Austria and the Comptoir d'Escompte of Paris, in combination with several capitalists. The Bonds will be issued at £13. 14*s*. 6*d*. each, with coupons attached, payable half-yearly, of the value of 9*s*. 11*d*. each, being at the rate of 5 per cent. per annum on the par value of £19. 17s. 0*d*. from the 1st December, 1865. They will be redeemed in 37 years by half-yearly drawings, to take place publicly, at the Austrian Embassy in Paris, on the 1st May and 1st November of each year. At each drawing an equal number of Bonds, viz. 9,928, will be withdrawn and paid off at par (£19. 17*s*. 0*d*.) with the half-yearly dividend. Find, from these data, the rate of interest at which the Austrian Government are thus borrowing. 48

1904. Let G′ denote the inverse of the centre of gravity G of any triangle ABC; H the equilateral hyperbola which passes through the points A, B, C, G; E the ellipse which touches the sides of the triangle at the points where they are intersected by the lines AG′, BG′, CG′; show that the nine-point circle of the triangle touches the common tangents of the two curves H and E. 106

1905. If $\frac{d^2y}{dx^2} + ax^m y = f(m)$, show that the solution of $f(m) = 0$ may be made to depend upon that of $f\left(-\frac{m}{m+1}\right) = 0$. Show also that the solution of $\frac{d^2y}{dx^2} + \phi\left(\pm\frac{1}{x}\right)\frac{1}{x^4}y = 0$ may be made to depend upon that of $\frac{d^2y}{dx^2} + \phi(x)\,y = 0$, the function ϕ denoting any function whatever. 33

1921. n counters are marked with the numbers $1, 3, 5 \ldots . (2n-1)$ on both faces, and a person taking one is to have as many shillings as the number marked on the counter; find the value of a person's expectation who takes one after m have been drawn. Also find the value of the expectation when one side only is marked as above, the other sides being marked with the even numbers up to $2n$. 32

1926. Find general methods of investigating similar series to Euler's and Machin's used for the calculation of π, and prove the relation

$$\frac{\pi}{4} = \left\{\frac{1}{\pi} - \frac{1}{3\pi^3} + \frac{1}{5\pi^5} - \&c.\right\} + \left\{\frac{\pi-1}{\pi+1} - \frac{(\pi-1)^3}{3(\pi+1)^3} + \&c.\right\}.$$ 94

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1931. Find the stationary tangents (or tangents at the inflexions) of the nodal cubic
$$x(y-z)^2+y(z-x)^2+z(x-y)^2=0. \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad 18$$

1932. Démontrer la formule
$$\int_{-\infty}^{\infty} f\left(x-\frac{1}{x}\right)dx=\int_{-\infty}^{\infty} f(x)\,dx \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad 19$$

1936. If the image of a plane conic formed by refraction through a thin lense be another plane conic, show that the cone subtending both conics from the centre of the lens will be a cone of revolution. 20

1939. Prove that the two feet-perpendicular lines corresponding to any point on the circumscribing circle of a pair of *diametral* triangles intersect at right angles on an ellipse tangential to the six sides. [ABC, A′B′C′ are called *diametral* triangles when AA′, BB′, CC′ intersect in the centre of the common circumscribing circle.].. 104

1940. Given that $1.2.3\ldots x\ (x \text{ inf.})\ \sqrt{(2\pi)}\, x^{x+\frac{1}{2}} \epsilon^{-x}$, prove that
$$m(m+n)\ldots\{m+(x-1)n\}\ (x \text{ inf.}) = \frac{\sqrt{(2\pi)}}{\Gamma\left(\frac{m}{n}\right)}\, n^x x^{x+\frac{m}{n}-\frac{1}{2}}\, \epsilon^{-x}. \quad 70$$

1941. AA′B′B is a quadrilateral inscribed in a conic. Two tangents PP′, QQ′ meet the diagonals AB′, A′B in the points P, P′, Q, Q′ respectively. Show that a conic can be described so as to touch AA′, BB′, and also pass through the four points P, P′, Q, Q′. 56

1942. If a, b are the semi-axes of an ellipse, and ϕ, ϕ' the eccentric angles of two points P, Q on the curve; prove that the area of the parallelogram formed by the tangents at the ends of the diameters through P and Q is $4ab \operatorname{cosec}(\phi'-\phi)$. 21

1949. Find the conic of five-pointic intersection at any point of the cuspidal cubic $y^3=x^2z$ 57

1951. To find the trilinear equation to the circle whose centre is at the point $(\alpha', \beta', \gamma')$ and whose radius is r. 22

1953. If three coordinate planes be inclined to one another at angles of 120°, prove that the relation between the direction-cosines of a line is
$$(\cos\lambda+\cos\mu)^2+(\cos\mu+\cos\nu)^2+(\cos\nu+\cos\lambda)^2=\tfrac{4}{3}. \quad\ldots\ldots \quad 22$$

1954. Let ABC be any triangle; α, β, γ the middle points of BC, CA, AB, respectively; and O the centre of the circumscribing circle. Draw Oα, Oβ, Oγ; and produce these lines to A′, B′, C′, making OA′=2Oα, OB′=2Oβ, OC′=2Oγ; and let O′ be the centre of the circle circumscribing A′B′C′. Then, if OO′ be bisected in M, the circle (M) to radius Mα will be tangential to the *thirty-two* inscribed and escribed circles of the system of triangles ABC, A′B′C′, AO′B, BO′C, CO′A, A′OB′, B′OC′, C′OA′. 25

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1955. Show that $\dfrac{a^{4b+1}-a}{30}$ is an integer. 26

1958. Let P denote a point in the plane of a given triangle ABC; α, β, γ the feet of the perpendiculars drawn from P upon the sides BC, CA, AB. Then if the triangle $\alpha\beta\gamma$ is homologous with the given one ABC, the locus of P is a cubic which passes through (1) the angular points of the triangle; (2) the centres of the inscribed and escribed circles; (3) the point of intersection of the three perpendiculars; (4) the centre, O, of the circumscribing circle; (5) the points L, M, N, where the radii AO, BO, CO produced meet this circle again. Moreover, if P′ denote the inverse of P with respect to the sides of the given triangle, show that P′ also lies on the cubic-locus in question. 59

1960. Required the curve bounding a sun-dial, whose plane is given in position, such that the length of its arc, measured from the 12 o'clock hour line, may be proportional to the time from 12 o'clock. 28

1963. Show that the equation of a circle on the line joining (x', y') and (x'', y'') as diameter is

$$(x-x')\ (x-x'')+(y-y')\ (y-y'')=0. \quad ....\ 47$$

1967. If an ellipse whose foci are G and G′ be inscribed in a plane triangle ABC, and if one focus G be the centroid of the triangle, prove (1) that the distances of the other focus G′ from the sides of the triangle are as the lengths of those sides, (2) that the sum of the squares of those distances is a minimum, and (3) that the distances of G′ from the angles are $G'A = a' \operatorname{cosec} A \div (\cot A + \cot B + \cot C)$, &c. &c., where a', b', c' are the distances from A, B, C to the middle points of BC, CA, AB. 28

1969. In two given great circles of a sphere intersecting at O are taken respectively two points P and Q, the arc joining which is of given length; prove that S, H, two fixed points, and M a fixed line, in a plane may be found such that, for all positions of the arc PQ, a point M in the fixed line may be found satisfying the equations

$$SM \pm HM = \sin OP,\ SM \mp HM = \sin OQ. \quad ....\ 67$$

1970. Find the conditions in order that the conics

$$U = (a,\ b,\ c,\ f,\ g,\ h)\ (x, y, z)^2 = 0,$$
$$U' = (a', b', c', f', g', h')\ (x, y, z)^2 = 0,$$

may have double contact. 99

1972. Find the envelope and locus of centres of a system of circles which intercept constant lengths on a fixed line and a fixed circle. 99

1974. If $x^2 = y^2 + z^2$, show that it can furnish no numerical formula which is not contained in the identical equation

$$\left(a+\frac{1}{a}\right)^2 \equiv \left(a-\frac{1}{a}\right)^2 + 4. \quad ....\ 39$$

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1975. Prove the following construction for finding the point whose trilinear coordinates are the reciprocals of those of a given point P. From P draw Pα, Pβ, Pγ perpendicular to the sides of the triangle of reference ABC; and let O be the centre of the circle round $\alpha\beta\gamma$: join PO and produce it to P′, making OP′ = PO: then P′ is the point required. 43

1981. If from the angular points of any triangle ABC, lines be drawn making the same constant angles with the adjacent sides, *four* triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, $A_4B_4C_4$, will be formed, which possess the following properties.

(1) The above triangles are all similar to each other and to the triangle ABC; (2) if circles be described about A_1CA, B_1AB, C_1BC, they will meet in a point P; (3) circles described about A_3BA, B_3CB, C_3AC, will meet in another point P_1; (4) if O_1, O_2, O_3 be the centres of the circles in (2) and O_4, O_5, O_6 the centres of those in (3), then the triangles $O_1O_2O_3$, $O_4O_5O_6$, ABC are similar, and the triangle $O_1O_2O_3$ is equal to $O_4O_5O_6$. 110

1983. Find the roots of the equation $x^3+px^2+qx+r=0$, the ratio of any two of these roots being given. 40

1986. Given four points on a circle: it is required to show that the "polar centres" of the four triangles that can be formed from them lie on another circle of equal radius. 80

1990. (1.) Prove that the locus of one set of foci of all the conics that touch a given circle at two given points, is another circle passing through those points and the centre of the given circle.

(2.) Prove that the three points in which a circular cubic is cut by any transversal are the foci of a Cartesian oval passing through the four foci of the cubic.

(3.) Prove that a circular cubic is the locus of one set of foci of all the conics that can be drawn through four points lying in a circle.

(4.) Prove that, if a circle and straight line be cut by any transversal in three points, these will be the foci of one of a system of Cartesian ovals having double contact with one another at two fixed points. 35, 70, 88, 100

1994. Two circles have double internal contact with an ellipse, and a third circle passes through the four points of contact. If t, t', T be the tangents from any point on the ellipse to these three circles, prove that $T^2=t\,t'$. 71

1996. If four circles $A=0$, $B=0$, $C=0$, $D=0$ are mutually orthotomic, the square of the radius of a circle $lA+mB+nC+sD=0$ is $(l^2r_1^2+m^2r_2^2+n^2r_3^2+s^2r_4^2)\div(l+m+n+s)^2$, where r_1, r_2, r_3, r_4 are the radii of A, B, C, D. 74

2001. If r and r_1 be the radii of two circles each having double contact with a conic, the former passing through the centre of the conic, and the latter through one of the foci; prove that $r:r_1=a:2b$. 75

2002. If a quadrilateral be inscribed in a conic, and either pair of opposite sides BA, CD, be produced to meet in E; then the

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line joining the point E with G, the intersection of the tangents at A and D, will pass through the intersections of the diagonals of the quadrilateral. 93

2006. Two conics expressed by their general equations touch one another at the origin; find the condition that they should touch each other in one other point. 95

2114. Prove that tan cos (θ) =

$$\frac{\frac{\cos\theta}{1}-\frac{\cos 3\theta}{1.2.3}+\frac{\cos 5\theta}{1.2.3.4.5}-\dots}{1-\frac{\cos 2\theta}{1.2}+\frac{\cos 4\theta}{1.2.3.4}-\dots}=\frac{\frac{\sin 2\theta}{1\ 2}-\frac{\sin 4\theta}{1.2.3.4}+\frac{\sin 6\theta}{1.2.3.4.5.6}-\dots}{\frac{\sin\theta}{1}-\frac{\sin 3\theta}{1.2.3}+\frac{\sin 5\theta}{1.2.3.4.5}-\dots}$$ 64

2222. Prove that if $2x+1$ be any *prime* number, and if a and b be any two unequal whole numbers, not multiples of $2x+1$, then either a^x+b^x or a^x-b^x will be exactly divisible by $2x+1$. ... 106

2226. If a conic have double contact with two other conics, prove that the chords of intersection of these two conics both pass through the intersection of the chords of contact of the two conics with the first conic. 104

2230. Soit F (x) un polynome qui reste positif pour toutes les valeurs réelles de la variable; il en sera de même du polynome suivant:

$$\Phi(x) = F(x) + aF'(x) + a^2F''(x) + a^3F'''(x) + \&c.$$

quel que soit la constante a.

Et si le polynome F (x) est quelconque, la plus grande racine de l'équation $\Phi(x) = 0$ sera inférieure à la plus grande des racines de $F(x) = 0$, si la constante a est positive.

Plus généralement, soit $\Theta(x) = 1 + ax + \beta x^2 + \&c. = 0$, une equation dont toutes les racines *sont réelles et positives;* si l'on fait

$$\frac{1}{\Theta(x)} = 1 + ax + bx^2 + cx^3 + \&c.$$

la plus grande racine réelle de

$$\Phi(x) = F(x) + a\,F'(x) + b\,F''(x) + c\,F'''(x) + \&c.$$

sera audessous de la plus grande racine réelle de $F(x) = 0$, et si le polynome $F(x)$ est positif quel que soit x, il en sera de même de $\Theta(x)$. Seulement alors il suffit que toutes les racines de $\Theta(x) = 0$ soient réelles, sans être toutes positives. 101

2239. If a draughtman lie on one of the intersections of the board, show that the sum of the arcs bounding the white sectors is always equal to the sum of the arcs bounding the black sectors. 111

2240. If a conic be described about a triangle ABC, and tangents at (B, C), (C, A), (A, B) meet respectively in G, H, K: then, if D, E, F be any three points in BC, CA, AB such that AD, BE, CF are concurrent, the three lines GD, HE, KF will also be concurrent. 90

2257. A line, fixed in length and position, is cut at two variable points into three segments the sum of whose squares is constant; required the locus of the vertex of the equilateral triangle described on the middle segment as base. 103

Unsolved Questions.

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1843. (Proposed by N'IMPORTE.)—Three points being taken at random within a circle, find the chance that the circle drawn through them will lie wholly within the given circle.

1849. (Proposed by Professor SYLVESTER.)—Two points are taken at random, one on each side of a given diameter of a circle: find the probability that the chord drawn through them shall not exceed a given length.

1850. (Proposed by Professor SYLVESTER.)—In a parabolic segment cut off by a line perpendicular to the axis two points are taken; find the mean value of such of the chords drawn through them as cut the segment in two points.

1860. (Proposed by M. W. CROFTON, B.A.)—Two points being taken at random on the *perimeter* of a rectangle, find the chance of the distance between them being less than a given length. (N.B. The same law will be found to hold here as in the *Note* at the end of the solution of Quest. 132; *Reprint*, Vol. IV. p. 86.)

1869. (Proposed by W. S. BURNSIDE, B.A.)—In the analysis of the fundamental formulæ for the addition and subtraction of elliptic functions, prove geometrically that the following transformations are legitimate; viz., first change k into k^{-1}, and then change $\sin \lambda mp$ into $k \sin \lambda mp$.

1870. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.)—Suppose there to be a promiscuous series of numbers in which the decimals have been cut off and adjusted to the nearest units of the terminal figures. Then on summing $n-1$ numbers arbitrarily taken, if from the arithmetical mean of the probabilities of the accumulated error respectively exceeding $e-\frac{1}{2}$ and $e+\frac{1}{2}$, of those units, there be subtracted $\frac{e}{n} \times$ probability of the error falling between $e-\frac{1}{2}$ and $e+\frac{1}{2}$, the difference will be the probability that the error will exceed e units when a summation includes n values.

1882. (Proposed by M. GARDINER.)—Defining the area of a curvilinear plane figure as by polar coordinates in the integral calculus, prove the following general theorem. If, at one extremity, a variable line of constant length touch in every position a plane closed re-entering curve of any form consisting of m right and of n left loops, the area of the figure described by the other extremity in the course of a complete revolution differs from that of the original figure by $(m-n)$ times the area of a circle whose radius is equal to the constant length of the line.

1884. (Proposed by FELSINIUS.)—Donner une démonstration géométrique du théorème suivant dû à M. Salmon. Si l'on a dans un plan deux systèmes de points, tels qu' à chaque point du premier système correspondent m points de l'autre, et à chaque point de ce dernier correspondent n points du premier, et si à une droite quelconque du premier système cor-

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respond dans l'autre une courbe be l'ordre l, le nombre des points doubles sera $l+m+n$.

1889. (Proposed by Chief Justice COCKLE, F.R.S.)—Solve, finitely, the differential equation $\frac{dy}{dx}+by^2=\frac{a}{x^6}$, or differential together with multiple of square of dependent variable equal to multiple of inverse sixth power of independent variable.

1891. (Proposed by Professor MANNHEIM.)—The tangents at a double point A on a quartic cut the curve again in B and C so that BC is a double tangent. Prove that $\frac{\rho_1\rho_1'}{\rho_2\rho_2'}=\frac{AB^3}{AC^3}$, where ρ_1, ρ_2 are the radii of curvature at A to the branches which touch AB, AC, and ρ_1', ρ_2' are the radii of curvature at B and C.

1907. (Proposed by W. K. CLIFFORD.)—Three ternary quadrics, U, V, W, break up into linear factors 1, 1′; 2, 2′; 3, 3′, respectively. Prove that
$\square(U,V,W)\equiv 123.1'2'3'+1'23.12'3'+12'3.1'23'+123'.1'2'3$, where $\square(U, V, W)$ is the coefficient of $6\lambda\mu\nu$ in the discriminant of $\lambda U+\mu V+\nu W$, and 123 means the determinant formed with the coefficients of the linear factors 1, 2, 3. Required the developments and interpretations.

1910. (Proposed by Professor SYLVESTER.)—AB is a given straight line upon which four points are taken at random. Find the chance that their anharmonic ratio (estimated by the quotient of the whole into the middle by the product of the extreme segments) shall exceed a given quantity.

1912. (Proposed by Professor MANNHEIM.)—If a and b be the points of contact, with a curve of the third class, of a double tangent; and if this tangent be intersected in m, n, p by the three tangents to the curve which can be drawn from any point in the plane, then $\frac{am\,.\,an\,.\,ap}{bm\,.\,bn\,.\,bp}=\frac{\rho_a}{\rho_b}$, where ρ_a, ρ_b are the radii of curvature at the points a and b.

1914. (Proposed by W. DAVIS.)—Find, to 10 decimals, *all* the roots of the equation $x^7+28x^4=480$.

1917. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.)—Suppose the surface of a sphere to be made up of an indefinite number of points, and straight lines to be drawn through every two of those points, and determine the law of density of this mass of lines as depending on the distance from the centre of the sphere.

1918. (Proposed by W. K. CLIFFORD.)—It is known that the conic of five-pointic contact at any point A of a cubic meets the curve again in a point B constructed by joining the point A to its second tangential; let this point be called the *conic tangential* of A. Then the conic tangential of B will be the second conic tangential of A, and so on. Show how, having given the conic tangential of any order, and also the line tangential

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of any order, we can construct for the original point A by the ruler alone.

1924. (Proposed by M. W. Crofton, B.A.)—Prove that the arc of a Cartesian oval, at any point P, is equally inclined to the straight line from P through any one focus, and to the circular arc from P through the two other foci. [The theorem in Question 1906 follows as a corollary from this.]

1933. (Proposed by C. Taylor, M.A.)—If α, β be the eccentric angles of two adjacent vertices of a polygon inscribed in a fixed ellipse, and if the polygon envelope a fixed confocal, then will $\Sigma\cos(\alpha-\beta) =$ a constant.

1934. (Proposed by M. W. Crofton, B.A.)—Two bags contain m and n balls respectively; an arbitrary number is drawn from each (0 being considered a number): find the chance of the total number drawn being equal to any assigned integer, from 0 to $m+n$. [See Note to Solution of Question 1321, *Reprint*, Vol. IV., p. 86.]

1945. (Proposed by C. W. Merrifield, F.R.S.)—To find a rectangular parallelepiped, such that its edges, the diagonals of its faces, and the diagonals of the solid, shall all be integral. [The Proposer states that he has not been able to solve this problem, and would therefore be glad of a solution, or of a proof that it is impossible.]

1952. (Proposed by E. Prouhet.)—Si l'on désigne par X_p^n le nombre total de manières dont un polygone convexe de n côtés peut être décomposé en p parties au moyen de $(p-1)$ diagonales qui ne se coupent pas dans l'intérieur du polygone, on a

$$X_p^n = \frac{1}{p}\cdot\frac{n(n+1)(n+2)\dots(n+p-2)}{1.2.3\dots(p-1)}\cdot\frac{(n-3)(n-4)\dots(n-p-1)}{1.2.3\dots(p-1)}.$$

Pour $n=p-2$ on retrouve la formule d'Euler démontrée dans le Journal de Liouville, tom. III. (1838), p. 505 et 547.

1956. (Proposed by R. Ball, M.A.)—If in any binary quantic $(a_0, a_1 \dots\dots a_n)(x, 1)^n$, or $F(x)$, x be changed into $\lambda + \dfrac{nF(\lambda)}{nx'F(\lambda)-F'(\lambda)}$, and the result be cleared of fractions by multiplying it by $\{nx'F(\lambda) - F'(\lambda)\}^n$; show that the coefficient of every power of x' in the expression thus obtained is a covariant of $F(\lambda)$.

1962. (Proposed by W. K. Clifford.)—Required the *characteristics* of the system of conics having five-pointic contact with a curve of order m and class n.

1966. (Proposed by S. Bills.)—Can the expression

$$(p^2+q^2)^4+64p^2q^2(p^2-q^2)^2$$

ever be a square (p and q being both rational), except when $p=q$?

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

NOTE ON THE FORMULA FOR QUADRATURES.

By W. S. B. WOOLHOUSE, F.R.A.S.

The June Number of the *Educational Times* contains an article "On Approximation to a Curvilinear Area," by Professor DE MORGAN, in which he gives the results of a determination of the numerical coefficients appertaining to the formula usually employed in calculating quadratures. The coefficients, according to a method founded on the "Calculus of Operations," are those of the symbolic expansion of $\left\{\log(1+\Delta)\right\}^{-1}$. They may therefore be found to any number of terms by working out, by long division, the reciprocal of the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+$ &c. Or, if C_n denote the nth coefficient, it may be computed, from those which precede it, by the formula

$$C_n = \frac{C_{n-1}}{2} - \frac{C_{n-2}}{3} + \frac{C_{n-3}}{4} - \ldots \pm \frac{C_1(=1)}{n}.$$

In allusion to these arithmetical values, Professor DE MORGAN's interesting Note concludes as follows:—

"The coefficients of $\left\{\log(1+\Delta)\right\}^{-1}$, so far as usually given, are

$$1,\ \tfrac{1}{2},\ -\tfrac{1}{12},\ \tfrac{1}{24},\ -\tfrac{19}{720},\ \tfrac{3}{160},\ -\tfrac{863}{60480}.$$

The three next, whether ever before printed, I know not, are

$$\tfrac{275}{24192},\ -\tfrac{33953}{3628800},\ \tfrac{8183}{1036800}."$$

The object of the present communication is merely to point out that, by an investigation conducted on the principles of ordinary Algebra, the whole of these coefficients had already been determined in my paper on "Summation," printed in the *Journal of the Institute of Actuaries*. See Vol. XI., p. 309, where, with respect to a series of equidistant ordinates V_0, V_1, V_2, &c., it is found that "the curvilinear area bounded by V_0 and V_n =

$$(V_0+V_1 \ldots + V_n) - \tfrac{1}{2}(V_n+V_0) - \tfrac{1}{12}(a'-a) - \tfrac{1}{24}(b'+b)$$
$$- \tfrac{19}{720}(c'-c) - \tfrac{3}{160}(d'+d) - \tfrac{863}{60480}(e'-e) - \tfrac{275}{24192}(f'+f)$$
$$- \tfrac{33953}{3628800}(g'-g) - \tfrac{8183}{1036800}(h'+h) \ \ldots\ldots\ldots\ldots (B)."$$

It will be perceived that this formula includes all the coefficients stated by Professor De Morgan. Moreover, the general formula (A), expressing the result of the summation of an interpolated finite series of values, is carried out to the same order of differences, that is, to the eighth order. It is, however, to the integration formula (B) only that reference is needed on this occasion, and the practical inference to be drawn is, that the identity of Professor De Morgan's numerical coefficients with those previously determined by a process so entirely different may be accepted as a satisfactory proof of their accuracy.

1808. (Proposed by A. G.)—Décomposer un nombre triangulaire en d'autres nombres triangulaires, dans toutes les manières possibles.

Solution by Samuel Bills.

Let a be the root of a given triangular number, and let x and y be the roots of two triangular numbers into which it is to be decomposed.

By the nature of triangular numbers, we must have

$$x^2+x+y^2+y = a^2+a. \quad \text{Assume} \quad y = a-\frac{r}{s}x, \text{ then}$$

$$x^2+x+a^2-2a\frac{r}{s}x+\frac{r^2}{s^2}x^2+a-\frac{r}{s}x = a^2+a;$$

therefore $$x = \frac{s(2ra+r-s)}{r^2+s^2};$$

where r and s may be any numbers that will make x and y positive integers. Let $a = 6$, and take $r = 1$, and $s = 3$; then $x = 3$ and $y = 5$; thus the triangular number 21 is decomposed into two triangular numbers 6 and 15. If $a = 8$, $r = 1$, and $s = 2$; therefore $x = 6$ and $y = 5$.

We might now proceed to decompose these latter numbers, where possible, as, for instance, 6 into 3 and 3. We should thus find *all* the triangular numbers of which the given one is composed. In precisely the same manner we may decompose pentagonal numbers into other pentagonal numbers, or, indeed, any m-gonal number into other m-gonal numbers.

1931. (Proposed by Professor Cayley.)—Find the stationary tangents (or tangents at the inflexions) of the nodal cubic

$$x(y-z)^2+y(z-x)^2+z(x-y)^2 = 0.$$

Solution by the Proposer.

The equation may be transformed into the form

$$(-8x+y+z)^{\frac{1}{3}}+(x-8y+z)^{\frac{1}{3}}+(x+y-8z)^{\frac{1}{3}} = 0,$$

and it thence follows immediately that the stationary tangents are the lines

$$-8x+y+z = 0, \quad x-8y+z = 0, \quad x+y-8z = 0,$$

respectively, and that the three points of contact, or inflexions, are the intersections of these lines with the line $x+y+z=0$.

In fact, writing

$$X = kx+y+z,\quad Y = x+ky+z,\quad Z = x+y+kz,$$

we have identically

$$\begin{aligned}
&(X+Y+Z)^3-27XYZ \\
&=(k+2)^3(x+y+z)^3-27(kx+y+z)(x+ky+z)(x+y+kz) \\
&=(x^3+y^3+z^3)\left\{(k+2)^3-27k\right\} \\
&+3(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y)\left\{(k+2)^3-9(k^2+k+1)\right\} \\
&+\qquad 3xyz\left\{2(k+2)^3-9(k^3+3k+2)\right\} \\
&=(k-1)^2(k+8)(x^3+y^3+z^3)+3(k-1)^3(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y) \\
&\qquad -3(k-1)^2(7k+2)xyz.
\end{aligned}$$

Hence, writing $k=-8$, we have

$$\begin{aligned}
(X+Y+Z)^3-27XYZ &= -2187\left\{yz^2+y^2z+zx^2+z^2x+xy^2+x^2y-6xyz\right\} \\
&= -2187\left\{x(y-z)^2+y(z-x)^2+z(x-y)^2\right\}.
\end{aligned}$$

The equation of the given curve is therefore

$$(X+Y+Z)^3-27XYZ=0,\ \text{or}\ X^{\frac{1}{3}}+Y^{\frac{1}{3}}+Z^{\frac{1}{3}}=0,$$

where of course X, Y, Z have the values

$$X=-8x+y+z,\quad Y=x-8y+z,\quad Z=x+y-8z.$$

1932. (Proposed by E. Prouhet.)—Démontrer la formule

$$\int_{-\infty}^{\infty} f\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

Solution by the Proposer.

1. Soit d'abord $\phi(x)$ une fonction paire de x, c'est-à-dire une fonction telle que $\phi(x)=\phi(-x)$. Si l'on pose $z=\dfrac{1}{x}$, on aura

$$\begin{aligned}
\int_0^{\infty}\phi\left(x-\frac{1}{x}\right)dx &= \int_{\infty}^{0}\phi\left(\frac{1}{z}-z\right)d\,\frac{1}{z} \\
&= \int_{\infty}^{0}\phi\left(\frac{1}{z}-z\right)d\left(\frac{1}{z}-z\right)+\int_{\infty}^{0}\phi\left(\frac{1}{z}-z\right)dz.
\end{aligned}$$

Posons maintenant $\dfrac{1}{z}-z=u$, nous aurons pour $z=\infty$, $u=-\infty$, et $u=\infty$ pour $z=0$.

Donc $$\int_0^\infty \phi\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^\infty \phi(u)\,du - \int_0^\infty \phi\left(z-\frac{1}{z}\right)dz,$$

ou $$2\int_0 \phi\left(x-\frac{1}{x}\right)dx = \int_{-\infty} \phi(u)\,du.$$

Mais on a évidemment, puisque $\phi(x)$ est une fonction paire,

$$\int_{-\infty}^\infty \phi\left(x-\frac{1}{x}\right)dx = 2\int_0^\infty \phi\left(x-\frac{1}{x}\right)dx,$$

et dans le second membre de la précédente égalité on peut remplacer u par x;

donc $$\int_{-\infty}^\infty \phi\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^\infty \phi(x)\,dx \quad \ldots\ldots\ldots\ldots\ldots (1).$$

ce qui démontre la proposition pour le cas d'une fonction paire.

2. Si $\psi(x)$ est une fonction impaire, on aura

$$\int_{-\infty}^\infty \psi\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^\infty \psi(x)\,dx \quad \ldots\ldots\ldots\ldots\ldots (2),$$

car chaque membre est évidemment nul.

3. Si $f(x)$ est une fonction quelconque, on peut écrire

$$f(x) = \phi(x) + \psi(x),$$

$\phi(x)$ étant une fonction paire, et $\psi(x)$ une fonction impaire. En effet, il suffit de poser

$$\phi(x) = \tfrac{1}{2}\left\{f(x)+f(-x)\right\}, \quad \psi(x) = \tfrac{1}{2}\left\{f(x)-f(-x)\right\};$$

or si l'on ajoute les équations (1) et (2), on a

$$\int_{-\infty}^\infty \left\{\phi\left(x-\frac{1}{x}\right)+\psi\left(x-\frac{1}{x}\right)\right\}dx = \int_{-\infty}^\infty \left\{\phi(x)+\psi(x)\right\}dx.$$

ou bien $$\int_{-\infty}^\infty f\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^\infty f(x)\,dx \quad \ldots\ldots\ldots\ldots\ldots (3).$$

Application. $\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}$. Donc

$$\int_{-\infty}^\infty e^{-\left(x-\frac{1}{x}\right)^2}dx = \sqrt{\pi}, \quad \text{ou} \quad \int_{-\infty}^\infty e^{-\left(x^2+\frac{1}{x^2}\right)}dx = e^{-2}\sqrt{\pi}.$$

1936. (Proposed by the Rev. R. TOWNSEND, M.A.)—If the image of a plane conic formed by refraction through a thin lens be another plane conic, show that the cone subtending both conics from the centre of the lens will be a cone of revolution.

Solution by the PROPOSER.

The image of a plane by refraction through a thin lens being a quadric of revolution, one of whose foci is the centre of the lens; and every plane section of a quadric of revolution determining at each focus of the surface a cone of revolution; therefore, &c.

1942. (Proposed by CANTAB.)—If a, b are the semi-axes of an ellipse, and ϕ, ϕ' the eccentric angles of two points P, Q on the curve; prove that the area of the parallelogram formed by the tangents at the ends of the diameters through P and Q is $4ab$ cosec $(\phi'-\phi)$.

Solution by J. MCDOWELL, M.A.; S. W. BROMFIELD; *and others.*

1. If m, n be two adjacent sides of a parallelogram, α the contained angle, and m', n' the perpendiculars on m, n, respectively, from the opposite sides, then (A) its area is $\text{A} = m'n' \operatorname{cosec} \alpha$.

For $\text{A}^2 = (mn \sin \alpha)^2 = (mm')^2 = (nn')^2 = mnm'n' = mn \sin \alpha \,.\, m'n' \operatorname{cosec} \alpha$
$= \text{A} \,.\, m'n' \operatorname{cosec} \alpha$.

2. If P, Q and P′, Q′ be corresponding points on an ellipse and its auxiliary circle, then it is known that the chords PQ, P′Q′ will meet in a point T on the major axis; and also that tangents at P and P′ will meet in the same point on the major axis; hence if the auxiliary circle be turned about the major axis of the ellipse through an angle θ such that $\cos \theta = \dfrac{b}{a}$, the orthogonal projection of the circle and of P′Q′ on the plane of the ellipse will coincide with the ellipse and PQ respectively; also the tangent at P′ will be projected into the tangent at P.

3. If A be the area of any polygon inscribed in an ellipse or circumcumscribed about it, and A′ the area of the corresponding polygon connected with the auxiliary circle, it is now evident that

$$\text{A} = \text{A}' \,.\, \frac{b}{a}.$$

4. The theorem in the question is now easily proved. For the parallelogram circumscribing the auxiliary circle at the ends of diameters corresponding to the diameters of the ellipse through P and Q, clearly has the perpendicular distances between its pairs of opposite sides each $= 2a$, and one of its angles $= \phi'-\phi$, therefore its area $= 4a^2 \operatorname{cosec} (\phi'-\phi)$. Hence the area of the parallelogram formed by tangents at the extremities of P and Q is

$$\frac{b}{a} \,.\, 4a^2 \operatorname{cosec} (\phi'-\phi), \text{ or } 4ab \operatorname{cosec} (\phi'-\phi).$$

[Another proof is given by Mr. THOMSON, in the *Reprint*, Vol. III., p. 26.]

1953. (Proposed by C. W. MERRIFIELD, F.R.S.)—If three coordinate planes be inclined to one another at angles of 120°, prove that the relation between the direction-cosines of a line is

$$(\cos\lambda+\cos\mu)^2+(\cos\mu+\cos\nu)^2+(\cos\nu+\cos\lambda)^2=\tfrac{4}{3}.$$

Solution by H. TOMLINSON; S. W. BROMFIELD; REV. J. L. KITCHIN, M.A.; T. J. SANDERSON, B.A.; J. DALE; R. TUCKER, M.A.; *and many others.*

If α, β, γ be the angles between the coordinate axes, we have the known relation (see Frost's *Solid Geometry*, Art. 29),

$$\cos^2\lambda\sin^2\alpha+\cos^2\mu\sin^2\beta+\cos^2\nu\sin^2\gamma+2\cos\mu\cos\nu(\cos\beta\cos\gamma-\cos\alpha)$$
$$+2\cos\nu\cos\lambda(\cos\gamma\cos\alpha-\cos\beta)+2\cos\lambda\cos\mu(\cos\alpha\cos\beta-\cos\gamma)$$
$$=1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma\ ..\ (1).$$

Now if we imagine a sphere to be drawn round the origin as centre, its intersections with the coordinate planes will form a spherical triangle whose angles are each 120° (the angles between these *planes*), and whose sides will measure the angles α β, γ; hence we find $\cos\alpha=\cos\beta=\cos\gamma=-\frac{1}{2}$.

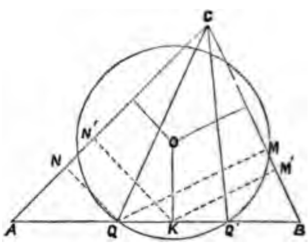
The relation (1) thus becomes, in the case proposed,

$$\cos^2\lambda+\cos^2\mu+\cos^2\nu+\cos\mu\cos\nu+\cos\nu\cos\lambda+\cos\lambda\cos\mu=\tfrac{2}{3},$$

or $$(\cos\lambda+\cos\mu)^2+(\cos\mu+\cos\nu)^2+(\cos\nu+\cos\lambda)^2=\tfrac{4}{3}.$$

1951. (Proposed by Professor WHITWORTH, M.A.)—To find the trilinear equation to the circle whose centre is at the point $(\alpha', \beta', \gamma')$ and whose radius is r.

I. *Solution by* T. J. SANDERSON, B.A.



Let ABC be the triangle of reference, O the point $(\alpha', \beta', \gamma')$ and centre of circle. The trilinear equation of the circle must be of the form

$$A\alpha^2+B\beta^2+C\gamma^2+2D\beta\gamma+2E\gamma\alpha+2F\alpha\beta=0\ ......\ (1);$$

and if in this equation we make

$$\gamma=0\ ....\ (2),$$

we have a quadratic in $\frac{\alpha}{\beta}$ denoting two straight lines passing through the intersections of (1) and (2), and also (since they are homogeneous) both passing through C, *i.e.* the quadratic in $\frac{\alpha}{\beta}$ is the equation of the two straight lines CQ, CQ′ in the figure.

We proceed then to form this equation. The equation to CQ is

$$\frac{\alpha}{\beta} = \frac{QM}{QN} = \frac{QK\sin B + KM'}{KN' - QK\sin A} = \frac{\sqrt{(r^2-\gamma'^2)}\sin B + \alpha' + \gamma'\cos B}{\beta' + \gamma'\cos A - \sqrt{(r^2-\gamma'^2)}\sin A},$$

or $\alpha\left\{\beta' + \gamma'\cos A - \sqrt{(r^2-\gamma')}\sin A\right\} - \beta\left\{\alpha' + \gamma'\cos B + \sqrt{(r^2-\gamma'^2)}\sin B\right\}$
$= 0 \ldots\ldots\ldots\ldots (3).$

Similarly the equation to CQ′ will be

$\alpha\left\{\beta' + \gamma'\cos A + \sqrt{(r^2-\gamma'^2)}\sin A\right\} - \beta\left\{\alpha' + \gamma'\cos B - \sqrt{(r^2-\gamma'^2)}\sin B\right\}$
$= 0 \ldots\ldots\ldots\ldots (4).$

The equation to the two lines CQ, CQ′ is, multiplying (3) and (4) together,

$$\alpha^2(\beta'^2+\gamma'^2+2\beta'\gamma'\cos A - r^2\sin^2 A) + \beta^2(\gamma'^2+\alpha'^2+2\gamma'\alpha'\cos B - r^2\sin^2 B)$$
$$+ 2\alpha\beta\left\{(\gamma'^2 - r^2(\sin A\sin B - (\alpha'+\gamma'\cos B)(\beta'+\gamma'\cos A)\right\} = 0.$$

This, then, is the equation which we proposed to find; and we can now write down, by symmetry, the required equation to the circle, viz.,

$$\begin{aligned}
&\alpha^2(\beta'^2+\gamma'^2+2\beta'\gamma'\cos A - r^2\sin^2 A)\\
&+\beta^2(\gamma'^2+\alpha'^2+2\gamma'\alpha'\cos B - r^2\sin^2 B)\\
&+\gamma^2(\alpha'^2+\beta'^2+2\alpha'\beta'\cos C - r^2\sin^2 C)\\
&+2\beta\gamma\left\{(\alpha'^2-r^2)\sin B\sin C - (\beta'+\alpha'\cos C)(\gamma'+\alpha'\cos B)\right\}\\
&+2\gamma\alpha\left\{(\beta'^2-r^2)\sin C\sin A - (\gamma'+\beta'\cos A)(\alpha'+\beta'\cos C)\right\}\\
&+2\alpha\beta\left\{(\gamma'^2-r^2)\sin A\sin B - (\alpha'+\gamma'\cos B)(\beta'+\gamma'\cos A)\right\} = 0.
\end{aligned}$$

II. *Solution by* F. D. THOMSON, M.A.

The perpendicular distance from the point x', y', z' (trilinear coordinates) on the line $\lambda x + \mu y + \nu z = 0$ is

$$\frac{\lambda x' + \mu y' + \nu z'}{\sqrt{(\lambda^2+\mu^2+\nu^2-2\mu\nu\cos A - 2\nu\lambda\cos B - 2\lambda\mu\cos C)}}$$

Hence if $x'y'z'$ be the centre of the circle, and $\lambda x + \mu y + \nu z = 0$ be a tangent, we have

$$(\lambda x' + \mu y' + \nu z')^2 = r^2(\lambda^2+\mu^2+\nu^2-2\mu\nu\cos A - 2\nu\lambda\cos B - 2\lambda\mu\cos C) \ldots \text{(i)}.$$

Hence taking λ, μ, ν to be the *tangential* coordinates of the tangent, (i) may be regarded as the *tangential* equation to the circle.

Arranging terms, (i) becomes

$$\lambda^2(x'^2-r^2) + \mu^2(y'^2-r^2) + \nu^2(z'^2-r^2) + 2\mu\nu\left\{y'z' + r^2\cos A\right\} + \&c. = 0.$$

Hence, if the corresponding *trilinear* equation be

$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$, we have

$$\begin{aligned}
A &= (y'^2-r^2)(z'^2-r^2) - (y'z' + r^2\cos A)^2\\
&= r^2\left\{r^2\sin^2 A - (y'^2 + z'^2 + 2y'z'\cos A)\right\}
\end{aligned}$$

$B = \&c.$, $C = \&c.$;

$$\begin{aligned}
F &= (z'x' + r^2\cos B)(x'y' + r^2\cos C) - (x'^2 - r^2)(y'z' + r^2\cos A)\\
&= r^2\left[(r^2 - x'^2)\sin B\sin C + (y' + x\cos C)(z' + x\cos B)\right]
\end{aligned}$$

$G = \&c.$, $H = \&c.$;

therefore the equation to the circle in trilinear coordinates is as given in the foregoing solution.

III. *Solution by* W. H. LAVERTY; H. TOMLINSON; *and others.*

We know that if d be the distance between two points $(\alpha\beta\gamma)$ and $(\alpha'\beta'\gamma')$ and if A, B, C, represent respectively the three determinants

$$\begin{vmatrix} \beta & \gamma \\ \beta' & \gamma' \end{vmatrix}, \quad \begin{vmatrix} \gamma & \alpha \\ \gamma' & \alpha' \end{vmatrix}, \quad \begin{vmatrix} \alpha & \beta \\ \alpha' & \beta' \end{vmatrix},$$

then

$$d = \frac{abc}{(2\Delta)^2}\left\{\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2 - 2(\mathrm{BC}\cos\mathrm{A} + \mathrm{CA}\cos\mathrm{B} + \mathrm{AB}\cos\mathrm{C})\right\}^{\frac{1}{2}}.$$

Hence in the circle in question we have

$$\frac{r^2 . (2\Delta)^4}{a^2b^2c^2} = \Sigma(\beta\gamma' - \beta'\gamma)^2 - 2\Sigma\left\{(\alpha\alpha'\beta'\gamma - \alpha^2\beta'\gamma' - \alpha'^2\beta\gamma - \alpha\alpha'\beta\gamma')\cos\mathrm{A}\right\}$$

therefore

$$\Sigma\left\{\alpha^2(\beta'^2 + \gamma'^2 + 2\beta'\gamma'\cos\mathrm{A})\right\} + 2\Sigma\left\{\beta\gamma(\alpha'^2\cos\mathrm{A} - \beta'\gamma' - \alpha'\beta'\cos\mathrm{B} - \gamma'\alpha'\cos\mathrm{C})\right\}$$

$$= \frac{r^2 . (2\Delta)^2 . (2\Delta)^2}{a^2b^2c^2}$$

$$= \frac{r^2 . b^2c^2 . \sin^2\mathrm{A}}{a^2b^2c^2}(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 + 2bc\beta\gamma + 2ca\gamma\alpha + 2ab\alpha\beta)$$

$$= r^2\left\{\alpha^2\sin^2\mathrm{A} + \beta^2\sin^2\mathrm{B} + \gamma^2\sin^2\mathrm{C} + 2\Sigma(\beta\gamma\sin\mathrm{B}\sin\mathrm{C})\right\},$$

and the equation becomes

$$\Sigma\left\{\alpha^2(\beta'^2 + \gamma'^2 + 2\beta'\gamma'\cos\mathrm{A} - r^2\sin^2\mathrm{A})\right\} + 2\Sigma\left\{\beta\gamma\left[(\alpha'^2 - r^2)\sin\mathrm{B}\sin\mathrm{C} - (\beta' + \alpha'\cos\mathrm{C})(\gamma' + \alpha'\cos\mathrm{B})\right]\right\} = 0.$$

1862. (Proposed by R. WARREN, B.A.) — Determine a system of values for (x, y, z), functions of (α, β, γ), and satisfying identically the equation $x^3 + y^3 + z^3 - 3xyz = (\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma)^2$.

Solution by S. ROBERTS, M.A.; REV. J. L. KITCHIN, M.A.; *and others.*

Let θ be an imaginary cube root of unity, then the given equation may be written in the form

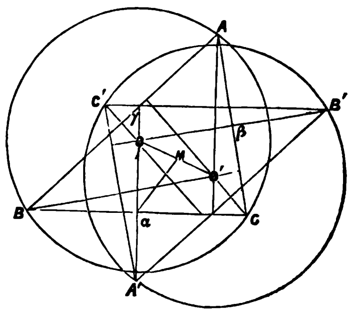
$$(x + y + z)(x + \theta y + \theta z)(x + \theta^2 y + \theta z)$$
$$= \left\{(\alpha + \beta + \gamma)(\alpha + \theta\beta + \theta^2\gamma)(\alpha + \theta^2\beta + \theta\gamma)\right\}^2 = k_1^2k_2^2k_3^2,$$

giving $3x = k_1^2 + k_2^2 + k_3^2$, $3y = k_1^2 + \theta^2k_2^2 + \theta k_3^2$, $3z = k_1^2 + \theta k_2^2 + \theta^2 k_3^2$. Having determined a set, we have six such sets by permutation.

Also if (x, y, z) satisfy the equation, $(x, \theta y, \theta^2 z)$ with their permutations also satisfy it.

1954. (Proposed by T. T. WILKINSON, F.R.A.S.)—Let ABC be any triangle; α, β, γ the middle points of BC, CA, AB, respectively; and O the centre of the circumscribing circle. Draw $O\alpha$, $O\beta$, $O\gamma$; and produce these lines to A′, B′, C′, making $OA' = 2O\alpha$, $OB' = 2O\beta$, $OC' = 2O\gamma$; and let O′ be the centre of the circle circumscribing A′B′C′. Then, if OO′ be bisected in M, the circle (M) to radius $M\alpha$ will be tangential to the *thirty-two* inscribed and escribed circles of the system of triangles ABC, A′B′C′, AO′B, BO′C, CO′A, A′OB′, B′OC′, C′OA′.

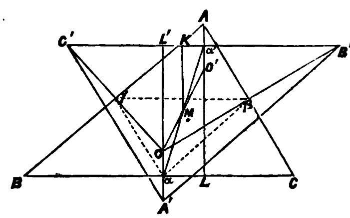
I. *Solution by* J. DALE.



Drawing the figure, it is easy to show that the triangle A′B′C′ is equal in all respects to ABC (see *Reprint*, Vol. I., pp. 6, 7), and so situated that O is the intersection of the perpendiculars of A′B′C′, while O′ is the intersection of the perpendiculars of ABC.

The circle (M) to radius ($M\alpha$) is the nine-point circle of every triangle in the system, and consequently by a known theorem tangential to the 32 inscribed and escribed circles of the 8 triangles ABC, A′B′C′, AO′B, BO′C, CO′A, A′OB′, B′OC′, C′OA′.

II. *Solution by* T. J. SANDERSON, B.A.



Since α, β, γ are the middle points of the sides, the triangle $\alpha\beta\gamma$ is similar to the triangle ABC, and of half its linear dimensions. Again, since α, β, γ are the middle points of the lines OA′, OB′, OC′, the triangle A′B′C′ is similar to the triangle $\alpha\beta\gamma$ and of *double* its linear dimensions. Hence the triangle A′B′C′ is similar and equal to the triangle ABC, but oppositely placed, homologous sides being parallel and M the centre of symmetry.

Produce A′O to meet C′B′ in L′, and A′L′ being perpendicular to BC is also perpendicular to B′C′. Draw MK perpendicular to C′B′. Then because $OM = O'M$, therefore $L'K = K\alpha'$ and the circle with centre M and radius $M\alpha'$ will therefore pass through L′; and it also passes through α, the middle point of OA′, therefore it is the nine-point circle of the triangle A′B′C; and similarly it is the nine-point circle of the triangle ABC. Therefore it passes through the points α, β, γ, α, β', γ', L, M, N, L′, M′, N′; (L, M, N, L′, M′, N′ are the feet of perpendiculars not all marked in the figure).

Again, because it passes through the points α', γ, and L, therefore it is the nine-point circle of the triangle AO′B, and similarly of the triangles BO′C, CO′A, A′OB′, B′OC′, C′OA′.

Now it is a known property of the nine-point circle of any triangle, that it touches the inscribed and three escribed circles of that triangle. Hence the circle with centre M and radius Mα touches the 32 inscribed and escribed circles of the 8 triangles ABC, A′B′C′, AO′B, BO′C, CO′A, A′OB′, B′OC′, C′OA′.

1955. (Proposed by C. M. INGLEBY, LL.D.)—Show that

$$\frac{a^{4b+1}-a}{30} \text{ is an integer.}$$

Solution by the REV. J. BLISSARD.

This Question, put in an extended form, may be expressed as follows:— Let $p_1, p_2 \ldots . p_n$ be prime numbers, all differing from each other; and let M be the least common multiple of the quantities $p_1-1, p_2-1, \ldots . p_n-1$. Required to prove that $\dfrac{a(a^{Mb}-1)}{p_1 p_2 \ldots p_n}$ is integral.

Proof.—By Fermat's Theorem, if p is a prime number, and if a does not contain p as factor, then $\dfrac{a^{p-1}-1}{p}$ is integral;

herefore $a^{p-1}-1$ is of the form mp (m integral), and $a^{p-1}=1+mp$;

$\therefore a^{(p-1)b}=(1+mp)^b$, and is of the form $1+m'p$, $\therefore \dfrac{a^{(p-1)b}-1}{p}$ is integral,

i.e., $\dfrac{a^x-1}{p}$ is integral if x contains $p-1$ as factor. Hence, generally, $\dfrac{a^x-1}{p_1 p_2 \ldots . p_n}$ is integral if x contains all the quantities $p_1-1, p_2-1, \ldots . p_n-1$ as factors, *i.e.*, if $x = \mathrm{M}b$, provided a does not contain any of the quantities $p_1, p_2, \ldots . p_n$ as factors. Hence $\dfrac{a(a^{Mb}-1)}{p_1 p_2 \ldots . p_n}$ must be integral, a being any positive integer; since, if a contains any of the p numbers as factors, $a^{Mb}-1$ must contain the rest. The following are examples:—

$$\frac{a(a^b-1)}{2},\ \frac{a(a^{2b}-1)}{2.3},\ \frac{a(a^{4b}-1)}{2.3.5} \text{ (the case supposed),}$$

$$\frac{a(a^{6b}-1)}{2.3.7},\ \frac{a(a^{10b}-1)}{2.3.5.11},\ \frac{a(a^{12b}-1)}{2.3.5.7.13} \text{ are all integral.}$$

COR.—If r of the p numbers, viz., $p_1, p_2, \ldots . p_r$, are repeated, and if M is the least common multiple of $p_1-1, p_2-1, \ldots . p_r-1$; then, from above,

$$\frac{a^2(a^{Mb}-1)(a^{M'b'}-1)}{p_1^2 p_2^2 \ldots . p_r^2 . p_{r+1} \ldots p_n} \text{ is integral; and so on generally.}$$

II. *Solution by* Dr. Booth, F.R.S.; H. Tomlinson; Rev. J. L. Kitchin, M.A.; M. Collins, B.A.; H. Murphy; *the* Proposer; *and others.*

The expression $\frac{a^{4x+1}-a}{30}$ may be put under the form

$$\frac{(a^x-1)\,(a^x+1)\,(a^{2x}+1)}{30a^{x-1}}.$$

Now as all numbers are of the forms $5n$, $(5n \pm 1)$, or $(5n \pm 2)$, and as the first three factors in the numerator are manifestly divisible by 2 and 3, unless they are also divisible by 5 it will be easy to show that $a^{2x}+1$ is divisible by 5.

For the forms $5n$ and $(5n \pm 1)$ being excluded by supposition, $a^{2x}+1=(5n \pm 2)^2+1=25n^2 \pm 20n+5$, which is divisible therefore by 5.

Hence 2, 3 and 5 are factors of the numerator; and if they should be factors of a^x they must also be factors of a, for a^x can contain no prime factors such as 2, 3, or 5 that are not also contained in a.

Hence $a(a^x-1)\,(a^x+1)\,(a^{2x}+1)$ is divisible by 30 as well as

$$a^x(a^x-1)(a^x+1)(a^{2x}+1);$$

or $\frac{a^{5x}-a^x}{30}$ and $\frac{a^{4x+1}-a}{30}$ are both integers.

III. *Solution by* W. H. Laverty; T. J. Sanderson, B.A.; S. Bills; S. W. Bromfield; R. Tucker, M.A.; *and others.*

To show generally that $\frac{a^{4b+d}-a^{4c+d}}{30}$ is an integer.

We know that, whatever be the last digit in any number, that number will still end in the same digit when raised to the $(4x+1)$th power.

Now $$\frac{a^{4b+d}-a^{4c+d}}{30}=\frac{a^{d-1}}{30}\left(a^{4b+1}-a^{4c+1}\right)$$

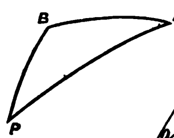
and both the numbers in the bracket will end in the same digit, therefore the difference of the two will end in a cipher and will therefore be divisible by 10. Again, a must be of one of the forms $3x-1$, $3x$, or $3x+1$; if of the second of these, the expression in the numerator is evidently divisible by 3; if of either of the others, we have

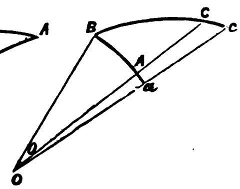
$$\frac{a^{4b+d}-a^{4c+d}}{30}=\frac{a^d}{30}\left\{(3x \pm 1)^{4b}-(3x \pm 1)^{4c}\right\}$$ and the "unities" in the expression under the bracket will cancel out, leaving the rest divisible by 3; therefore the whole expression in the numerator is divisible by 3×10, or 30.

Let now $d=1$; $c=0$; and we see that $\frac{a^{4b+1}-a}{30}$ is an integer.

1960. (Proposed by E. FITZGERALD.)—Required the curve bounding a sun-dial, whose plane is given in position, such that the length of its arc, measured from the 12 o'clock hour line, may be proportional to the time from 12 o'clock.

Solution by MATTHEW COLLINS, B.A.





Let P be the pole, PB $= a$ the meridian, AB $= \theta$ on the plane of the dial, O the point or centre where the line or style OP, *parallel to the earth's axis*, meets the plane of the dial, BCc the required curve, BC $=$ C, C$c = d$C, AOB $= \theta$, AO$a = d\theta$. The spherical triangle APB gives $\cot\theta \sin a = \cot \mathrm{P} \sin \mathrm{B} + \cos a \cos \mathrm{B}$, which differentiated gives $\sin a \operatorname{cosec}^2\theta \,.\, d\theta = \sin \mathrm{B} \operatorname{cosec}^2 \mathrm{P} \,.\, d\mathrm{P}$. Now, by the Question, we must have $\mathrm{C} = m \,.\, \mathrm{P}$, therefore $m^2 (d\mathrm{P})^2 = (d\mathrm{C})^2 = (dr)^2 + (rd\theta)^2$. By eliminating P and dP from this last equation by means of our first two equations, we find a differential equation containing only dr and $d\theta$, whose integral will be the equation of the required curve.

1967. (Proposed by M. COLLINS, B.A.)—If an ellipse whose foci are G and G′ be inscribed in a plane triangle ABC, and if one focus G be the centroid of the triangle, prove (1) that the distances of the other focus G′ from the sides of the triangle are as the lengths of those sides, (2) that the sum of the squares of those distances is a minimum, and (3) that the distances of G′ from the angles are $\mathrm{G'A} = a' \operatorname{cosec} \mathrm{A} \div (\cot \mathrm{A} + \cot \mathrm{B} + \cot \mathrm{C})$, &c. &c., where a', b', c' are the distances from A, B, C to the middle points of BC, CA, AB.

Solution by J. DALE; T. J. SANDERSON, B.A.; REV. R. H. WRIGHT, M.A.; R. TUCKER, M.A.; *and many others.*

1. Let (x, y, z), (x', y', z') be the trilinear coordinates of G and G′; then, by well known properties,

$$xx' = yy' = zz' = (\text{semi-axis minor})^2, \quad \text{and} \quad ax = by = cz;$$

dividing each term in the first series by the corresponding term in the second, we have $x' : a = y' : b = z' : c$, which proves the theorem (1).

2. Suppose it is required to find the trilinear coordinates of a point within a triangle such that the sum of the squares of these coordinates may be a minimum. We have

$$\phi = x'^2 + y'^2 + z'^2 = \text{minimum}, \quad \text{and} \quad ax' + by' + cz' = 2\Delta;$$

differentiating, $x'\,dx' + y'\,dy' + z'\,dz' = 0, \quad a\,dx' + b\,dy' + c\,dz' = 0;$

whence we find $x' : a = y' : b = z' : c$; and, as this is the point found in (1), the theorem (2) is thereby proved.

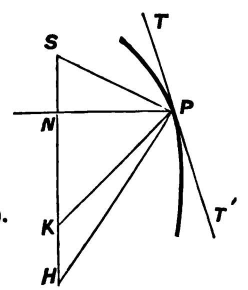
3. From the coordinates of G and G′, it appears that GA, GB, GC make the same angles with AB, AC; BC, BA; CA, CB, that G′A, G′B, G′C make with AC, AB; BA, BC; CB, CA. Hence we readily find

$$\frac{G'A}{a' \operatorname{cosec} A} = \frac{G'B}{b' \operatorname{cosec} B} = \frac{G'C}{c' \operatorname{cosec} C} = \frac{2x'}{a} = \frac{2y'}{b} = \frac{2z'}{c} = \frac{4\Delta}{a^2+b^2+c^2}$$
$$= \frac{2\Delta}{bc \cos A + ca \cos B + ab \cos C} = \frac{1}{\cot A + \cot B + \cot C},$$

which proves the theorem (3).

1736. (Proposed by R. TUCKER, M.A.)—Any number of chords of a given fixed plane curve are drawn through a fixed point, and are traversed by a number of perfectly elastic particles, moving from rest at the fixed point; find the form of the curve, if, after impact on the perfectly smooth arc, all the particles pass through another fixed point vertically below the former.

Solution by the PROPOSER.



Let S. H, be the two fixed points, SP one of the chords, and TPT′ the tangent at P. Draw the horizontal line PN, and suppose the particle after impact to move initially at an $\angle$ NPK $(=\epsilon)$ with PN; then if SH $= c$, NK $= z$, SN $= x$, PN $= y$, $\angle$PSN$=\theta$, $\angle$HPN $= \iota$, we have

$$(y \sec \iota) \cos^2 \iota = 4\kappa \cos \epsilon \sin (\iota - \epsilon),$$

(by the theory of projectiles), and

$$\frac{\sin(\iota-\epsilon)}{\cos \iota} = \frac{c-x-z}{(y^2+z^2)^{\frac{1}{2}}};\ \therefore y^2+z^2 = 4x(c-x-z)\ (1).$$

Again, $\dfrac{z}{y} = \cot \text{PKN} = \cot(\theta - 2\psi)$

$$\left(\text{where} \quad \tan\psi = \frac{dy}{dx} = p\right)$$

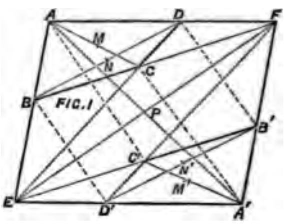
$$= \frac{\cot 2\psi \cot\theta + 1}{\cot 2\psi - \cot\theta} = \frac{x(1-p^2)+2py}{y(1-p^2)-2px} = -\frac{2x \mp (4cx-y^2)^{\frac{1}{2}}}{y}, \text{ by (1)};$$

$$\therefore p^2\left\{3xy \mp y(4cx-y^2)^{\frac{1}{2}}\right\} - 2p\left\{1 \pm x(4cx-y^2)^{\frac{1}{2}} + y^2 - 2x^2\right\}$$
$$- \left\{3xy \mp y(4cx-y^2)^{\frac{1}{2}}\right\} = 0.$$

The solution of this equation will give the curve required.

1592. (Proposed by N'IMPORTE.)—To prove that the middle points of the three diagonals of a complete quadrilateral are in the same straight line.

I. *Solution by* D. M. ANDERSON.



Let ABCDEF (Fig. 1) be a complete quadrilateral, and M, N, P, the middle points of its three diagonals AC, BD, EF. Complete the parallelogram AEA'F; draw EB', FD', intersecting in C', parallel to BF, DE, respectively, and let M', N' be the middle points of A'C', B'D'. Now the diagonals of each of the parallelograms BDB'D', ACA'C' obviously intersect in P; and therefore NN', MM', respectively parallel to their sides BD', A'C', and bisecting the aforesaid parallelograms, pass each through P. But if FD' and AE (produced) meet (say) in L, we have

$$LD' : LF = LE : LA, \text{ and } LC' : LF = LE : LB,$$

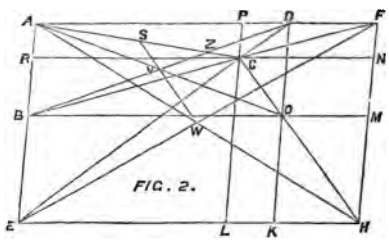
therefore $$LD' : LC' = LB : LA$$

and therefore BD' is parallel to AC'. Hence NN', MM' are parallel to each other, and passing each through one point P they must wholly coincide; that is to say, M, N, P are in one straight line.

COROLLARY.—From the above we see that if AEA'F is any parallelogram, and if EB', BF are parallel to each other, and *any* straight line FC'D' is drawn, then BD' is parallel to AC'.

[Mr. ANDERSON remarks that the parallelism of AC' and BD' may be otherwise proved, though not so briefly, by the aid of the *first* book of Euclid.]

II. *Solution by* MATTHEW COLLINS, B.A.



Let ABCDEF be the complete quadrilateral; AC, BD, EF being its three diagonals; and draw lines parallel to AE, AF, as in Fig. 2: then from the parallelograms ABMF, AEKD, we have, by Euc. I. 43, parallelogram AC = CM = CK, and therefore parallelogram NO = OL. Hence by Euc. I. 43 (*ex absurdo*), the points C, O, H are in the same straight line. Now as the diagonals of a parallelogram bisect each other, the middle points of BD, EF are the same as the middle points of AO, AH; but the middle points S, V, W of AC, AO, AH lie evidently in a straight line parallel to COH; hence the theorem is proved.

COR. 1. It follows, by projection, that if the three diagonals AC, BD, EF of a complete quadrilateral be cut by any straight line at P, Q, R, the harmonic conjugates of the points P, Q, R, taken on these diagonals, will be in the same straight line.

COR. 2. By polar reciprocation it follows that if A, B, C, D be four points and O any fifth point; and if through E (the point of intersection of AB and CD) we draw the straight line EO′, the harmonic conjugate of EO relative to EBA, ECD; and draw also the harmonic conjugates of FO and ZO, relative to the lines AD, BC (meeting in F), and to AC, BD (meeting in Z): then these three harmonic conjugates of EO, FO, ZO will meet in one point.

[The theorems in Mr. COLLINS's corollaries (the first of which includes that in the question) will furnish good exercises in the combined methods of tangential and trilinear coordinates. Putting (x, y, z) for tangential coordinates, the respective equations of the points A, B, C, E, F, D may be written

$$x = 0, y = 0, z = 0, lx + my = 0, my + nz = 0, lx + my + nz = 0 \ldots\ldots (1).$$

Moreover, the equations of three points (P, Q, R) and of their harmonic conjugates (P′, Q′, R′), which form on the diagonals AC, BD, EF the harmonic ranges (APCP′), (BQDQ′), (ERFR′), may be written

$$x \pm \lambda z = 0,\ (lx + my + nz) \pm \mu y = 0,\ (lx + my) \pm \nu (my + nz) = 0 \ldots . (2);$$

one set of signs applying for (P, Q, R) and the other two for their harmonic conjugates (P′, Q′, R′).

Now the points (P, Q, R) or (P′, Q′, R′) will be in a straight line according as the determinant

$$\begin{vmatrix} 1, & 0\ , & \pm\lambda \\ l, & m \pm \mu\ , & n \\ l, & m \pm \nu m\ , & \pm \nu n \end{vmatrix}$$

is zero for the upper or the lower signs; but this determinant, when expanded, becomes

$$m(\lambda\nu l - n) - \mu(\lambda l - \nu n)$$

for both sets of signs: hence, if it vanish for one set of signs, it will vanish also for the other set: if, therefore (P, Q, R) be in a straight line, (P′, Q′, R′) will also be in a straight line, which proves the theorem in Mr. COLLINS's first corollary.

The same equations, with a different interpretation, will furnish a proof of the theorem in the second corollary. For considering now (x, y, z) as trilinear coordinates, equations (1) denote the sides and diagonals (EA, AD, DE, AC, BD, BF) of a complete quadrilateral ABCDEF; and equations (2) are those of the harmonic pencils (E.AODO′), (F.BOAO′), (Z.CODO′); and the same determinant shows that if the lines EO, FO, ZO meet in a point O, their harmonic conjugates will also meet in a point O′, which proves the theorem in the second corollary.]

1613. (Proposed by R. TUCKER, M.A.)—Find the locus of the vertices of a system of similar ellipses described upon the diameters of a given similar ellipse as parameters; also find the envelope of the *other* parameters.

Solution by the PROPOSER; E. MCCORMICK; E. FITZGERALD; *and others.*

Take for axes the principal axes of the fixed ellipse; then, $2a'$ being the major axis of one of the ellipses, the locus of the vertices (ρ, ϕ) will be given by

$$\rho = a'(1 \pm e), \quad r = a'(1-e^2), \quad \phi = \tfrac{1}{2}\pi \pm \theta,$$

$$r^2 = \frac{a^2(1-e^2)}{1-e^2\cos^2\theta}; \therefore \rho^2 = \frac{a^2(1 \pm e)}{1 \mp e} \cdot \frac{1}{1-e^2\sin^2\phi},$$

the equations to two ellipses.

We have for equation to the other parameter

$$Y - X\tan\theta = -2a'e\sec\theta = -\frac{2ae\sec\theta}{(1-e^2)^{\frac{1}{2}}(1-e^2\cos^2\theta)^{\frac{1}{2}}}$$

and differentiating, $X = \dfrac{2ae\sin\theta\,(1-2e^2\cos^2\theta)}{(1-e^2)^{\frac{1}{2}}(1-e^2\cos^2\theta)^{\frac{3}{2}}}$;

hence the *radial* of the envelope is found to be

$$(1-e^2)^{\frac{1}{2}}(1-e^2\sin^2\theta)^{\frac{5}{2}}r = 2ae\left\{1+e^2-2e^2(2-e^2)\sin^2\theta\right\}.$$

1921. (Proposed by R. TUCKER, M.A.)—n counters are marked with the numbers 1, 3, 5....$(2n-1)$ on both faces, and a person taking one is to have as many shillings as the number marked on the counter; find the value of a person's expectation who takes one after m have been drawn. Also find the value of the expectation when one side only is marked as above, the other sides being marked with the even numbers up to $2n$.

Solution by S. BILLS; H. HOSKINS; *and others.*

Taking the question in a more general form; suppose the n counters to be marked with the numbers $a_1, a_2, a_3 .. a_n$ on one side, and with the numbers $b_1, b_2, b_3 ... b_n$ on the other side; and let it be required to find the value of the expectation of a person drawing the n counters in succession, the numbers representing shillings.

The total amount of the numbers on the n counters being $= \Sigma(a) + \Sigma(b)$, and the number of counters being n, the value of the expectation of drawing the first will be $\frac{1}{n}\left\{\Sigma(a)+\Sigma(b)\right\}$ shillings.

Again; after one has been drawn, since there will $n-1$ remaining, the value of the expectation of drawing the second will be

$$\frac{1}{1-n}\left\{\Sigma(a)+\Sigma(b) - \frac{\Sigma(a)+\Sigma(b)}{n}\right\} = \frac{1}{n}\left\{\Sigma(a)+\Sigma(b)\right\}$$

shillings, the same as for the first.

It thus appears that the value of the expectation of drawing every one in succession will be the same, and equal to $\frac{1}{n}\left\{\Sigma(a)+\Sigma(b)\right\}$.

Now, in the first part of the question, $\Sigma(a) = \Sigma(b) = n^2$, therefore the value required $= 2n$. In the second part $\Sigma(a) = n^2$ and $\Sigma(b) = n^2 + n$, therefore the value required $= 2n + 1$.

1905. (Proposed by Chief Justice COCKLE, F.R.S.)—

If $\frac{d^2y}{dx^2} + ax^m y = f(m)$, show that the solution of $f(m) = 0$ may be made to depend upon that of $f\left(-\frac{m}{m+1}\right) = 0$. Show also that the solution of $\frac{d^2y}{dx^2} + \phi\left(\pm\frac{1}{x}\right)\frac{1}{x^4}y = 0$ may be made to depend upon that of $\frac{d^2y}{dx^2} + \phi(x)\,y = 0$, the function ϕ denoting any function whatever.

Solution by the PROPOSER.

LEMMA.—If
$$\frac{d^2y}{dx^2} + ry = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$
be soluble, then also
$$\frac{d^2y}{dx^2} + \frac{2}{x}\cdot\frac{dy}{dx} + ry = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (2)$$
is soluble.

For if, in (1), we substitute $x\mathrm{Y}$ for y and divide by x, the result, after replacing Y by y, will be (2).

I. Divide (1) by r, differentiate, then multiply into r, and in the product replace $\frac{dy}{dx}$ by y. The result will be
$$\frac{d^2y}{dx^2} - \frac{1}{r}\cdot\frac{dr}{dx}\cdot\frac{dy}{dx} + ry = 0 \;\ldots\ldots\ldots\ldots\ldots (3).$$
Change the independent variable from x to t, and (3) becomes, after a slight reduction,
$$\frac{d^2y}{dt^2} - \left(\frac{1}{r}\cdot\frac{dr}{dt} + \frac{dt}{dx}\cdot\frac{d^2x}{dt^2}\right)\frac{dy}{dt} + r\left(\frac{dx}{dt}\right)^2 y = 0 \;\ldots\ldots (4).$$
Let
$$-\frac{1}{r}\cdot\frac{dr}{dt} - \frac{dt}{dx}\cdot\frac{d^2x}{dt^2} = \frac{c}{t} \;\ldots\ldots\ldots\ldots\ldots\ldots (5),$$
then, integrating and reducing, and denoting by G the arbitrary constant,
$$r\frac{dx}{dt} = \mathrm{G}t^{-c} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$
Next, put $r = ax^m$, and integrate again; we may say, making the second arbitrary constant zero,
$$\frac{ax^{m+1}}{m+1} = \frac{\mathrm{G}t^{1-c}}{1-c} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$
whence
$$x = \left(\frac{1+m}{1-c}\cdot\frac{\mathrm{G}}{a}\right)^{\frac{1}{m+1}}\cdot t^{\frac{1-c}{1+m}}, \text{ and } \frac{dx}{dt} = \left(\frac{\mathrm{G}}{a}\right)^{\frac{1}{m+1}}\cdot\left(\frac{1-m}{1-c}\right)^{\frac{-m}{m+1}}\cdot t^{-\frac{c+m}{1+m}} \;\ldots\ldots\ldots (8, 9),$$
also
$$r\left(\frac{dx}{dt}\right)^2 = ax^m\left(\frac{dx}{dt}\right)^2 = a^{-\frac{1}{m+1}}\cdot\mathrm{G}^{\frac{m+2}{m+1}}\left(\frac{1+m}{1-c}\right)^{-\frac{m}{m+1}}\cdot t^{-\frac{(m+2)c+m}{m+1}} \quad (10),$$
and if we make $\mathrm{G} = a\left(\frac{1+m}{1-c}\right)^{\frac{m}{m+2}}$, $k = -\frac{(m+2)c+m}{m+1}$ ...... (11, 12),

we shall have made the solution of $f(m) = 0$ depend upon that of

$$\frac{d^2y}{dt^2} + \frac{c}{t} \cdot \frac{dy}{dt} + at^k = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13),$$

in which I here suppose that c is not equal to unity.

In (11, 12) and (13), make $c=0$, then $k = -\frac{m}{m+1}$, and if in the reduced (13) we replace t by x, that equation becomes

$$\frac{d^2y}{dx^2} + ax^{-\frac{m}{m+1}} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (14).$$

In other words, we have made the solution of $f(m) = 0$ depend upon that of $f\left(-\frac{m}{m+1}\right) = 0$. The cases of $m = -1$ and $m = -2$ are exceptional.

SCHOLIUM.—The Lemma shows that connected solutions will be obtained when $c=2$, in which case $k = -\frac{3m+4}{m+1}$. Hence the solution of $f(m) = 0$ may also be made to depend upon that of $f\left(-\frac{3m+4}{m+1}\right) = 0$. Further, if the solutions of $f(m) = 0$ and $f\{\psi(m)\} = 0$, ψ being a functional symbol, are connected, then also the solutions of

$$f(m) = 0,\ f\{\psi(m)\} = 0,\ f\{\psi^2(m)\} = 0,\ \ldots\ f\{\psi^r(m)\} = 0, \text{ \&c.}$$

are connected, one with every other. Again, χ being another functional symbol, and $f(m) = 0$ and $f\{\chi(m)\} = 0$ being connected in solution, every equation of the system represented by

$$f\{\psi^r[\chi^q(m)]\} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (15),$$

wherein r and q may receive any integral values whatever, is connected with every other. And the same holds when, instead of two functional symbols ψ and χ, we have any number of such symbols with any indices. In (15) let $r=q=1$, $\chi(m) = -\frac{m}{m+1}$, and $\psi(m) = -\frac{3m+4}{m+1}$,

then $$f\{\psi[\chi(m)]\} = f(-m-4) = 0 \ldots\ldots\ldots\ldots (16)$$

is connected with $f(m) = 0$. This is a particular case of the theorem which I am about to demonstrate.

II. In virtue of the Lemma the solution of $\frac{d^2y}{dx^2} + \phi(x)y = 0 \quad \ldots (17)$

may be made to depend upon that of $\frac{d^2y}{dx^2} + \frac{2}{x} \cdot \frac{dy}{dx} + \phi(x)y = 0 \ldots (18)$.

Let $$x = -\frac{1}{t},\quad \therefore \frac{dx}{dt} = \frac{1}{t^2},\quad \therefore \frac{d^2x}{dt^2} = -\frac{2}{t^3} \ldots\ldots (19),$$

and change the independent variable from x to t. Then (18) becomes, after slight reductions, successively

$$\frac{d^2y}{dt^2} + \left(\frac{2}{x} \cdot \frac{dx}{dt} - \frac{dt}{dx} \cdot \frac{d^2x}{dt^2}\right) \frac{dy}{dt} + \phi(x) \left(\frac{dx}{dt}\right)^2 y = 0 \ldots (20),$$

and $$\frac{d^2y}{dt^2} + \left(-\frac{2}{t} + \frac{2}{t}\right)\frac{dy}{dt} + \phi\left(-\frac{1}{t}\right)\frac{1}{t^4}y = 0 \quad \text{........(21)},$$

or $$\frac{d^2y}{dt^2} + \phi\left(\frac{-1}{t}\right)\frac{1}{t^4}y = 0 \quad \text{..........................(22)}.$$

And since, whether we take $x = -\frac{1}{t}$ or $x = \frac{1}{t}$, the middle term of (21) alike vanishes ($xt \pm 1 = 0$ satisfies that condition), we see, on replacing t by x in (22), that the solutions of (17) and of

$$\frac{d^2y}{dx^2} + \phi\left(\pm\frac{1}{x}\right)\frac{1}{x^4}\cdot y = 0 \quad \text{...............(23)}$$

depend upon one another. Let $\phi(x) = ax^m$, then (17) becomes $f(m) = 0$, the solution of which is thus connected with that of $f(-m-4) = 0$. This verifies a result already given. Let $m = 2$, then the solution of $f(2) = 0$ is connected with that of $f(-2-4) = f(-6) = 0$. (Compare Questions 1854 and 1889.)

1990. (Proposed by Professor SYLVESTER.)—Prove that the three points in which a circular cubic is cut by any transversal are the foci of a Cartesian oval passing through the four foci of the cubic.

Solution by Professor CAYLEY.

Some preliminary explanations are required in regard to this remarkable theorem.

1. I call to mind that a circular cubic (or cubic through the two circular points at infinity) has 16 foci, which lie 4 together on 4 different circles; and that the property of 4 concyclic foci is that taking any three of them A, B, C, the distances of a point P of the curve from these three foci are connected by a linear relation $\lambda\,.\,AP + \mu\,.\,BP + \nu\,.\,CR = 0$, where $\lambda \pm \mu \pm \nu = 0$, or if as is more convenient the distances are considered as $\pm$, then where $\lambda + \mu + \nu = 0$. A circular cubic may be determined so as to satisfy 7 conditions; having a focus at a given point is 2 conditions; hence a circular cubic may be determined so as to pass through three given points, and to have as foci two given points.

2. A Cartesian, or bicircular cuspidal quartic (that is a quartic having a cusp at each of the circular points at infinity) has nine foci, but of these there are three which lie in a line with the centre of the Cartesian (or intersection of the cuspidal tangents), and which are preeminently the foci of the Cartesian. We may, therefore, say that the Cartesian has three foci, which foci lie in a line, the axis of the Cartesian. A Cartesian may be determined to satisfy 6 conditions; having a focus at a given point is 2 conditions; but having for foci three given points on a line is 5 conditions; and hence a Cartesian may be found having for foci three given points on a line, and passing through a given point; there are in fact two such Cartesians, intersecting at right angles at the given point.

3. The theorem at first sight appears impossible; for take any three points F, G, H in a line and any other point A; then, as just remarked, there are,

having F, G, H for foci and passing through A, two Cartesians. And we may draw through F, G, H, and with A for focus, a circular cubic depending upon two arbitrary parameters; the position of a second focus of the circular cubic is (on account of the two arbitrary parameters) *primâ facie* indeterminate; and this is confirmed by the remark that the circular cubic can actually be so determined as to have for focus an arbitrary point B; and yet the theorem in effect asserts that the foci concyclic with A, of the circular cubic, lie on one or other of the two Cartesians.

4. To explain this, it is to be remarked that the arbitrary point B is a focus which is either concyclic with A or else not concyclic with A. In the latter case, although B is arbitrary, yet the foci concyclic with A may and in fact do lie on one of the Cartesians; the difficulty is in the former case if it arises; viz., if we can describe a cubic through the points F, G, H in a line, and with A and B as *concyclic* foci; that is, if we can find a third focus C, such that the distances from A, B, C of a point P on the curve are connected by a relation $\lambda . AP+\mu . BP+\nu . CP=0$, where $\lambda+\mu+\nu=0$. It may be shown that this is in a sense possible, but that the resulting cubic is not a proper circular cubic, but is the cubic made up of the line FGH taken twice, and of the line infinity. To show this, since the required cubic passes through the points F, G, H we have

$$\begin{array}{l} \lambda . AF+\mu . BF+\nu . CF=0 \\ \lambda . AG+\mu . BG+\nu . CG=0 \\ \lambda . AH+\mu . BH+\nu . CH=0 \\ \lambda \quad +\mu \quad +\nu \quad =0 \end{array} \quad \text{and thence} \quad \left\| \begin{array}{cccc} AF, & AG, & AH, & 1 \\ BF, & BG, & BH, & 1 \\ CF, & CG, & CH, & 1 \end{array} \right\| = 0,$$

being two conditions for the determination of the position of the point C; these give CG, CH as linear functions of CF; the distances CF, CG, CH of the point C from the points F, G, H in the line FGH are connected by a quadratic equation, and hence substituting for CG, CH their values in terms of CF, we have a quadratic equation for CF; as the given conditions are satisfied when C coincides with A or with B, the roots of this equation are $CF=AF$ and $CF=BF$. But if $CF=AF$, then the linear relations give $CG=AG$ and $CH=AH$, that is, C is a point opposite to A in regard to the line FGH. And similarly if $CF=BF$, then C is a point opposite to B in regard to the line FGH. But C being opposite to A or B, the fourth concyclic focus D will be opposite to B or A; that is, the pairs A, B and C, D of concyclic foci lie symmetrically on opposite sides of the line FGH; this of course implies that the four points lie on a circle.

5. Taking $Y=0$ as the equation of the line FGH, $x^2+y^2-1=0$ as the equation of the circle through the four points A, B, C, D, then these lie on a proper cubic

$$(x^2+y^2+1)\,x+lx^2+ny^2=0$$

(not passing through the points F, G, H) and the four foci are given as the intersections with the circle $x^2+y^2-1=0$ of the pair of lines

$$x^2-2nx-nl=0.$$

But if we attempt to describe with the same four foci a cubic

$$(x^2+y^2+1)\,y+l'x^2+2m'xy+n'y^2=0,$$

then the foci are given as the intersections with the circle $x^2+y^2-1=0$ of the conic

$$y^2+2m'x-2l'y+m'^2-n'l'=0.$$

In order that these may coincide with the points (A, B, C, D) we must have

$$(x^2-2nx-nl)+(y^2+2m'x-2l'y+m'^2-n'l')=x^2+y^2-1;$$

that is, $\qquad m'=n, \quad l'=0, \quad -nl+n^2-n'l'=-1.$

The last equation is $n'l' = n^2+1-nl$, which, assuming that nl is not equal to n^2+1, [in this case the cubic $(x^2+y^2+1)x+lx^2+my^2=0$ would reduce itself to the line and conic $(x+n)\left(x^2+y^2+\dfrac{x}{n}\right)=0$], since $l'=0$, gives $n'=\infty$, and therefore the cubic

$$(x^2+y^2+1)\,y+l'x^2+2m'xy+n'y^2=0,$$

reduces itself to $y^2=0$, that is, the cubic in question reduces itself to the line F, G, H, twice repeated, and the line infinity.

6. The conclusion is that F, G, H being given points on a line, and A and B being any other given points, there is not any proper cubic passing through F, G, H and having A, B for concyclic foci: and the *primâ facie* objection to the truth of the theorem is thus removed.

7. Considering the points F, G, H on a line and the point A as given, it has been seen that there are *two* Cartesians through A with the foci F, G, H; and the theorem asserts that in the circular cubics through F, G, H with the focus A, the foci concyclic with A lie on one or other of the two Cartesians: there are consequently through F, G, H with the focus A two systems of circular cubics corresponding to the two Cartesians respectively, each system depending upon two arbitrary parameters. But if we attend only to one of the two Cartesians and to the corresponding system of cubics, then the Cartesian passes through the four foci of each cubic, and if (instead of taking as given the points F, G, H and the focus A) we take as given the four concyclic foci A, B, C, D of a cubic, the theorem asserts that we have through A, B, C, D a Cartesian depending on two arbitrary parameters (or having for its axis an arbitrary line), and such that the foci of the Cartesian are the points of intersection F, G, H of its axis with the cubic. And I proceed to the proof of the theorem in this form.

8. The equation of a circular cubic having four foci on the circle $x^2+y^2-1=0$ is

$$(x^2+y^2+1)\,(\mathrm{P}x+\mathrm{Q}y)+lx^2+2mxy+ny^2=0;$$

and this being so, the four foci are the intersections of the circle with the conic

$$(\mathrm{Q}x-\mathrm{P}y)^2+2\,(-n\mathrm{P}+m\mathrm{Q})\,x+2\,(m\mathrm{P}-l\mathrm{Q})\,y+m^2-nl=0.$$

9. The general equation of a Cartesian is

$$(x^2+y^2+2\mathrm{A}x+2\mathrm{B}y+\mathrm{C})^2+2\mathrm{D}x+2\mathrm{E}y+\mathrm{F}=0,$$

and by assuming for A, B, C, D, E, F the following values which contain the two arbitrary parameters a and θ, viz., by writing

$$2\mathrm{A}=\theta\mathrm{Q},\ 2\mathrm{B}=-\theta\mathrm{P},\ \mathrm{C}=a-1,\ \mathrm{D}=-n\theta^2\mathrm{P}+(m\theta^2-a\theta)\,\mathrm{Q},$$
$$\mathrm{E}=(m\theta^2+a\theta)\,\mathrm{P}-l\theta^2\mathrm{Q},\ \mathrm{F}=-a^2+\theta^2\,(m^2-nl),$$

we have the equation of a system (the selected one out of two systems) of Cartesians through the four foci; in fact, substituting the foregoing values, the equation of the Cartesian is

$$\left\{x^2+y^2+\theta\,(\mathrm{Q}x-\mathrm{P}y)+a-1\right\}^2-2a\theta\,(\mathrm{Q}x-\mathrm{P}y)$$
$$+2\theta^2\,(-n\mathrm{P}+m\mathrm{Q})\,x+2\theta^2\,(m\mathrm{P}-l\mathrm{Q})\,y-a^2+\theta^2\,(m^2-nl)=0,$$

and writing herein $x^2+y^2-1=0$, the equation reduces itself to

$$\theta^2\left\{(\mathrm{Q}x-\mathrm{P}y)^2+2\,(-n\mathrm{P}+m\mathrm{Q})\,x+2\,(m\mathrm{P}-l\mathrm{Q})\,y+m^2-nl\right\}=0,$$

verifying that the Cartesian passes through the four foci.

The coordinates of the centre of the Cartesian are $x=-\mathrm{A}$, $y=-\mathrm{B}$, and the equation of its axis is $\mathrm{E}\,(x+\mathrm{A})-\mathrm{D}\,(y+\mathrm{B})=0$; we have therefore to show that the points of intersection of this line with the cubic are the foci of the Cartesian.

10. To find where the line in question meets the cubic

$$(x^2+y^2+1)(Px+Qy)+lx^2+2mxy+ny^2=0,$$

writing in this equation $x=-A+D\Omega,\quad y=-B+E\Omega,$
we have for the determination of Ω the equation

$$\left\{A^2+B^2+1-2(AD+BE)\Omega+(D^2+E^2)\Omega^2\right\}\times$$
$$\left\{-AP-BQ+(DP+EQ)\Omega\right\}+(l,m,n)(-A+D\Omega,\ -B+E\Omega)^2=0,$$

or observing that we have $AP+BQ=0$, this equation becomes

$$\begin{aligned}
&(D^2+E^2)(DP+EQ)\Omega^3\\
+&\left\{-2(AD+BE)(DP+EQ)\qquad +lD^2+2mDE+nE^2\right\}\Omega^2\\
+&\left\{(A^2+B^2+1)(DP+EQ)\qquad -2lAD-2m(AE+BD)-2nBE\right\}\Omega\\
+&\left\{\qquad lA^2+2mAB+nB^2\right\}=0.
\end{aligned}$$

11. Substituting for A, B, D, E their values in terms of (P, Q, α, θ), we find

$$\begin{aligned}
DP+EQ &= -\theta^2(nP^2-2mPQ+lQ^2)\\
lA^2+2mAB+nB^2 &= \tfrac{1}{4}\theta^2(nP^2-2mPQ+lQ^2)\\
lAD+m(AE+BD)+nBE &= -\tfrac{1}{2}\alpha\theta^2(nP^2-2mPQ+lQ^2)\\
lD^2+2mDE+nE^2 &= \left((nl-m^2)\theta^4+\alpha^2\theta^2\right)(nP^2-2mPQ+lQ^2),
\end{aligned}$$

and substituting these values in the equation for Ω, the whole equation divides by $\theta^2(nP^2-2mPQ+lQ^2)$, and it then becomes

$$4(D^2+E^2)\Omega^3+4\left\{-2(AD+BE)-(nl-m^2)\theta^2-\alpha^2\right\}\Omega^2$$
$$+4\left\{A^2+B^2+1-\alpha\right\}\Omega-1=0,$$

or, putting for shortness

$$\begin{aligned}
C'&=C-A^2-B^2 &&= \alpha-1-A^2-B^2\\
F'&=F-2(AD+BE) &&=-\alpha^2-\theta^2(nl-m^2)-2(AD+BE),
\end{aligned}$$

the equation in Ω is

$$4(D^2+E^2)\Omega^3+4F'\Omega^2-4C'\Omega-1=0,$$

so that, Ω satisfying this equation, the intersections of the axis with the cubic are given by $x=-A+D\Omega,\ y=-B+E\Omega$.

12. The equation of the Cartesian, writing therein $x+A=u$ and $y+B=v$, and attending to the values of C′ and F′, is

$$(u^2+v^2+C')^2+2Du+2Ev+F'=0.$$

And to find the foci, writing in this equation $u+\rho,\ v+i\rho$ in place of u, v, we find $\left\{u^2+v^2+C'+2(u+vi)\rho\right\}^2+2(D+Ei)\rho+2Du+2Ev+F'=0,$
that is, $(u^2+v^2+C')^2+2Du+2Ev+F'$

$$+\left\{2(u+vi)(u^2+v^2+C')+D+Ei\right\}2\rho+4(u+vi)^2\rho^2=0.$$

Expressing that this equation in ρ has two equal roots, we find

$$4(u+vi)^2\left\{(u^2+v^2+C')^2+2Du+2Ev+F'\right\}$$
$$-\left\{2(u+vi)(u^2+v^2+C')+D+Ei\right\}^2=0,$$

that is, $4(2Du+2Ev+F')(u+vi)^2$

$$-4(u^2+v^2+C')(u+vi)(D+Ei)-(D+Ei)^2=0,$$

which equation is in fact the equation of the three tangents from one of the circular points at infinity. Writing it under the form $Y+Vi=0$, the nine foci of the Cartesian are given as the intersections of the two cubics $U=0$, $V=0$. But of these nine points, three, the foci that we are concerned with, lie on the axis, or line $Eu-Dv=0$; in fact, we have

$$\begin{aligned} U = {} & 4(u^2-v^2)(2Du+2Ev+F') \\ & -4(uD-vE)(u^2+v^2+C') \\ & -(D^2-E^2), \end{aligned} \qquad \begin{aligned} V = {} & 8uv(2Du+2Ev+F') \\ & -4(uE+vD)(u^2+v^2+C') \\ & -2DE; \end{aligned}$$

and hence $2DEU-(D^2-E^2)V$

$$=(Eu-Dv)\left\{8(Du+Ev)(2Du+2Ev+F')-4(D^2+E^2)(u^2+v^2+C')\right\}=0,$$

which shows that the nine points lie three of them on the line $Eu-Dv=0$, and the remaining six on the conic

$$2(Du+Ev)(2Du+2Ev+F')-(D^2+E^2)(u^2+v^2+C')=0.$$

13. We have thus the three foci given as the intersections of the axis $Eu-Dv=0$, with the cubic

$$\begin{aligned} U = 4(u^2-v^2)(2Du+2Ev+F') - 4(uD-vE)(u^2+v^2+C') \\ -(D^2-E^2)=0; \end{aligned}$$

or, writing in this last equation $u=D\Omega$, $v=E\Omega$, that is $x=-A+D\Omega$, $y=-B+E\Omega$, we have

$$u^2-v^2=(D^2-E^2)\Omega^2, \quad uD-vE=(D^2-E^2)\Omega.$$

The whole equation divides by (D^2-E^2), and omitting this factor, it is

$$4\Omega^2\left\{2(D^2+E^2)\Omega+F'\right\}-4\Omega\left\{(D^2+E^2)\Omega^2+C'\right\}-1=0,$$

that is
$$4(D^2+E^2)\Omega^3+4F'\Omega^2-4C'\Omega-1=0,$$

the same equation as the equation in Ω before obtained; that is the intersections of the cubic with the axis are the three foci of the Cartesian.

1974. (Proposed by C. M. Ingleby, LL.D.)—If $x^2=y^2+z^2$, show that it can furnish no numerical formula which is not contained in the identical equation $\left(a+\frac{1}{a}\right)^2 \equiv \left(a-\frac{1}{a}\right)^2+4$.

I. *Solution by* Samuel Bills.

Take the given formula and put it in the form $(x+y)(x-y)=z^2$. Now it is very obvious that the quantity $x+y$ can have no value but may be expressed in the form $x+y=az$, and then we should have $x-y=\frac{z}{a}$. From these two equations we find $x=\frac{1}{2}\left(a+\frac{1}{a}\right)z$, $y=\frac{1}{2}\left(a-\frac{1}{a}\right)z$. Substituting these results in the given formula, and dividing by z^2, we obtain $\left(a+\frac{1}{a}\right)^2=\left(a-\frac{1}{a}\right)^2+4$; hence the truth of the proposition.

II. *Solution by the* PROPOSER.

If $x^2 = y^2 + z^2$, multiply each term by 4 and divide by z^2; then $\left(\frac{2x}{z}\right)^2 = \left(\frac{2y}{z}\right)^2 + 4$; and putting $(x+y)+(x-y)$ and $(x+y)-(x-y)$ for $2x$ and $2y$ respectively, we get

$$\left(\frac{x+y}{z} + \frac{x-y}{z}\right)^2 = \left(\frac{x+y}{z} - \frac{x-y}{z}\right)^2 + 4,$$

and this becomes identical, if $\frac{x+y}{z}$ and $\frac{x-y}{z}$ are regarded as reciprocals of each other; *i.e.* if $\frac{x+y}{z} \cdot \frac{x-y}{z} = 1$; which is true if $x^2 - y^2 = z^2$. Therefore, if we put $\frac{x+y}{z} = a$ and $\frac{x-y}{z} = \frac{1}{a}$, we get $\left(a + \frac{1}{a}\right)^2 = \left(a - \frac{1}{a}\right)^2 + 4$; and for every triad of values for x, y, z, satisfying the original equation, there is a corresponding value of the ratio a, making the latter equation coincide with the former.

COR.—Every case is given by four sets of values for x, y, z, or a.

EX.—If $a = 2, \frac{1}{2}, 3$, or $\frac{1}{3}$, we get $5^2 = 4^2 + 3^2$. If $a = 4, \frac{1}{4}, \frac{5}{3}$ or $\frac{3}{5}$, we get $17^2 = 15^2 + 8^2$; and so forth. These values are determined by the conjugate equations, $a = \frac{x+y}{z}$, or $a = \frac{x+z}{y}$; and $\frac{1}{a}$ gives the same results as a, because $x^2 - (y^2 + z^2)$ is a reciprocal function of a.

1983. (Proposed by W. LEA.) — Find the roots of the equation $x^3 + px^2 + qx + r = 0$, the ratio of any two of these roots being given.

Solution by the REV. ROBERT HARLEY, F.R.S.

Let ρ be the given ratio; then, since

$$x^3 + px^2 + qx + r = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

we have

$$\rho^3 x^3 + p\rho^2 x^2 + q\rho x + r = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now $(1)-(2)$ and $\rho^3(1)-(2)$ give, after slight reduction,

$$(1+\rho+\rho^2)\,x^2 + p\,(1+\rho)\,x + q = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$p\rho^2 x^2 + q\rho\,(1+\rho)\,x + r\,(1+\rho+\rho^2) = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

And $p\rho^2(3) - (1+\rho+\rho^2)(4)$ gives

$$\rho\,(1+\rho)\left\{p^2\rho - q\,(1+\rho+\rho^2)\right\}x + pq\rho^2 - r\,(1+\rho+\rho^2)^2 = 0;$$

or

$$x = \frac{1}{\rho\,(1+\rho)} \cdot \frac{r\,(1+\rho+\rho^2)^2 - pq\rho^2}{p^2\rho - q\,(1+\rho+\rho^2)}, \text{ one root,}$$

therefore $\rho x = \dfrac{1}{1+\rho} \cdot \dfrac{r(1+\rho+\rho^2)-pq\rho^2}{p^2\rho-q(1+\rho+\rho^2)}$, a second root, and

$-p-(1+\rho)x = \dfrac{1}{\rho} \cdot \dfrac{-p^3\rho^2+pq\rho(1+\rho)^2-r(1+\rho+\rho^2)^2}{p^2\rho-q(1+\rho+\rho^2)}$, the third root.

NOTE.—These expressions may of course be exhibited in a variety of forms, because a relation exists between the ratio ρ and the coefficients p, q, r.

That relation, obtained by the elimination of x is as follows, viz.,

$$\begin{aligned} r^2\rho^6-(pqr-3r^2)\rho^5+(p^3r-5pqr+q^3+6r^2)\rho^4 \\ +(2p^3r-p^2q^2-6pqr+2q^3+7r^2)\rho^3 \\ +(p^3r-5pqr+q^3+6r^2)\rho^2-(pqr-3r^2)\rho+r^2=0, \end{aligned}$$

a recurring equation, which is, as might be expected, of the sixth degree in ρ. The six values of ρ, considered as a function of the roots x_1, x_2, x_3, are $\dfrac{x_1}{x_2}$, $\dfrac{x_1}{x_3}$, $\dfrac{x_2}{x_3}$, and their reciprocals.

The above equation may be put under the form

$$r^2\left(\rho+\frac{1}{\rho}\right)^3-(pqr-3r^2)\left(\rho+\frac{1}{\rho}\right)^2+(p^3r-5pqr+q^3+3r^2)\left(\rho+\frac{1}{\rho}\right)$$
$$+2p^3r-p^2q^2-4pqr+2q^3+r^2=0 \ldots\ldots\ldots\ldots\ldots (5).$$

The condition that (1) may have two equal roots, obtained from (5) by making $\rho+\dfrac{1}{\rho}=2$, is $4p^3r-p^2q^2-18pqr+4q^3+27r^2=0$,

the first member of which is (to a factor *près*) the discriminant of (1).

This is known to be true from other considerations.

II. *Solution by* R. BALL, M.A.

If k be the ratio of two roots α and β of the cubic $(a, b, c, d)(x, 1)^3 = 0$, we have the equations

$$\alpha-k\beta=0,\quad \alpha+\beta+\gamma+\frac{3b}{a}=0,\quad \alpha\beta+\alpha\gamma+\beta\gamma-\frac{3}{a}=0.$$

From the solution of these equations we obtain

$$\left.\begin{aligned} \alpha &= \frac{-6ck}{3b(1+k)\mp\sqrt{\{9b^2(1+k)^2-12ac(1+k+k^2)\}}} \\ \beta &= \frac{-6c}{3b(1+k)\mp\sqrt{\{9b^2(1+k)^2-12ac(1+k+k^2)\}}} \\ \gamma &= -3\frac{b}{a}+\frac{6c+6ck}{3b(1+k)\mp\sqrt{\{9b^2(1+k)^2-12ac(1+k+k^2)\}}} \end{aligned}\right\}\ldots(\mathrm{I}).$$

It remains to account for the ambiguity of sign. This arises from the circumstance that α, β, γ have not been determined to satisfy the cubic, but merely the three equations from which they were derived. To be the roots of the cubic they must further make $\alpha\beta\gamma=-\dfrac{d}{a}$, and that sign must be attached to the radical which will fulfil this condition. One, then, of the

signs must, if k be really the ratio of two roots, satisfy this; but, it may be inquired of what cubic are the three values of α, β, γ the roots, if the other sign be given to the radical? The answer is not difficult.

Let $(\alpha-k\beta)(\beta-k\alpha)(\alpha-k\gamma)(\gamma-k\alpha)(\beta-k\gamma)(\gamma-k\beta)$ be expanded and their values substituted for the symmetric functions: the result must equal zero, since we have assumed that k is the ratio of a pair of roots. If in the equation thus produced k be regarded as variable, the roots are the ratios of the roots of the given cubic. This equation may therefore be found by the elimination of α between

$$\begin{cases} \alpha = \dfrac{-6ck}{3b(1+k) \mp \sqrt{\{9b^2(1+k)^2 - 12ac(1+k+k^2)\}}} \\ a\alpha^3 + 3b\alpha^2 + 3c\alpha + d = 0, \end{cases}$$

or since it is reciprocal it may be obtained more easily by forming the symmetric functions which are its coefficients. The result is

$$\begin{aligned} &a^2d^2k^6 + (3a^2d^2 - 9abcd)k^5 + (6a^2d^2 - 45abcd + 27ac^3 + 27b^3d)k^4 \\ &+ (7a^2d^2 - 54abcd + 54ac^3 - 81b^2c^2 + 54b^3d)k^3 \\ &+ (6a^2d^2 - 45abcd + 27ac^3 + 27b^3d)k^2 + (3a^2d^2 - 9abcd)k + a^2d^2 = 0 \ldots\ldots (\text{II}). \end{aligned}$$

This is easily verified by the consideration that if the cubic have equal roots k must equal unity; but if in this k be made $= 1$, the result becomes

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd = 0,$$

the well known condition for equal roots.

Since, however, k is a given quantity a, b, c, d must be such as to satisfy (II); and since d enters in the second degree, there are two values of d which ($a, b,$ and c remaining the same) will accomplish this. One of these values for d must, of course, be that given in the cubic; call the other d'. Then we find that there are two cubics, namely,

$$(a, b, c, d)(x, 1)^3 = 0, \text{ and } (a, b, c, d')(x, 1)^3 = 0,$$

whose roots satisfy the system

$$\alpha - k\beta = 0, \ \alpha+\beta+\gamma+\frac{3b}{a} = 0, \ \beta\gamma+\gamma\alpha+\alpha\beta - \frac{3c}{a} = 0.$$

The upper sign then in the values (I) constitutes them the roots of one of these equations, the lower sign gives the roots of the other.

Suppose that k is not the ratio of two of the roots of $(a, b, c, d)(x, 1)^3 = 0$, of what cubics are the systems (I) the roots? Here k and d do not then fulfil the relation (II), but by solving (II) as a quadratic for d, two values d' and d'' are found, and the cubics required are

$$(a, b, c, d')(x, 1)^3 = 0, \qquad (a, b, c, d'')(x, 1)^3 = 0.$$

This remark may be stated generally. Suppose a rational and integral relation is given among the roots of a cubic $f(\alpha, \beta, \gamma) = 0$ in which the roots enter to the nth degree. If we can solve

$$f(\alpha, \beta, \gamma) = 0, \quad \alpha+\beta+\gamma+\frac{3b}{a} = 0, \quad \beta\gamma+\gamma\alpha+\alpha\beta - \frac{3c}{a} = 0,$$

$2n$ systems are found; one of these systems are roots of the given cubic: to what cubics do the others belong?

$f(\alpha, \beta, \gamma) \cdot f(\alpha, \gamma, \beta) \cdot f(\beta, \alpha, \gamma) \cdot f(\beta, \gamma, \alpha) \cdot f(\gamma, \alpha, \beta) \cdot f(\gamma, \beta, \alpha) = 0$:— evaluate this symmetric function in terms of the coefficients, the "weight"

of the result is $6n$; but the weight of d is 3, therefore d will enter in the power of $2n$, consequently $2n$ values of d will satisfy this result; if these values be $d', d'', d''', d'''^{\cdot 2n}$, the system of cubics is

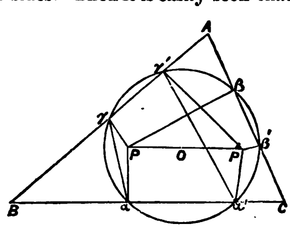
$(a, b, c, d')(x, 1)^3 = 0$, $(a, b, c, d'')(x, 1)^3 = 0$, $(a, b, c, d'''^{\cdot 2n})(x, 1)^3 = 0$;

one of which will, of course, be identical with the given cubic.

1975. (Proposed by F. D. THOMSON, M.A.)—Prove the following construction for finding the point whose trilinear coordinates are the reciprocals of those of a given point P. From P draw Pα, Pβ, Pγ perpendicular to the sides of the triangle of reference ABC; and let O be the centre of the circle round $\alpha\beta\gamma$: join PO and produce it to P$'$, making OP$'$ = PO: then P$'$ is the point required.

Solution by the PROPOSER.

Draw P$'\alpha'$, P$'\beta$, P$'\gamma'$ perpendicular to the sides. Then it is easily seen that the circle $\alpha\beta\gamma$ passes through $\alpha'\beta'\gamma'$. Join $\alpha\gamma$, $\alpha'\gamma'$. Then Bα . Bα' = Bγ . Bγ', therefore Bα : Bγ = Bγ' : Bα'. Hence the triangles B$\gamma\alpha$, B$\alpha'\gamma'$ are similar, therefore $\angle$ B$\gamma\alpha$ = B$\alpha'\gamma'$, and $\angle \alpha\gamma$P = $\gamma'\alpha'$P$'$: hence the triangle $\gamma\alpha$P is similar to $\alpha'\gamma'$P$'$, therefore γP : αP = α'P$'$: γ'P$'$; and similarly αP : βP = β'P$'$: α'P$'$; therefore αP . α'P$'$ = βP . β'P$'$ = γP . γ'P$'$, which proves the theorem.



[Mr. THOMSON remarks that the above was suggested by Mr. BESANT's paper in the *Messenger of Mathematics*, Vol. III., p. 222. See also the Solution of Question 1815, on page 19 of Vol. V. of the *Reprint*.

Mr. DALE proves the theorem by supposing the circle $\alpha\beta\gamma$ to cut BC, CA, AB in α', β', γ', and producing αP to meet the circumference in α''. Then the perpendicular to BC through α' obviously meets PO produced in P$'$, and α''P = α'P$'$; also αP . Pα'' = αP . α'P$'$ = R^2 − PO^2 (R being the radius of the circle $\alpha\beta\gamma$): and similarly it may be shown that the perpendiculars through β', γ' likewise pass through P$'$, and that

$$\alpha P \,.\, \alpha'P' = \beta P \,.\, \beta'P' = \gamma P \,.\, \gamma'P' = R^2 - PO^2.]$$

1896. (Proposed by Dr. WILSON.)—Show that the lines trisecting an angle of a triangle do not trisect the opposite side.

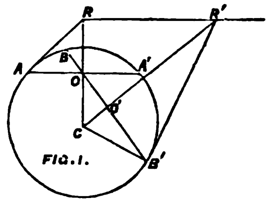
Solution by J. R. Allen; S. W. Bromfield; R. Tucker, M.A.; J. Dale; A. Renshaw; W. H. Laverty; J. H. Taylor, B.A.; *and many others.*

If the bisector of an angle of a triangle bisects the opposite side, the triangle is isosceles (by Euc. vi. 3) and the bisector perpendicular to the side. If therefore the two trisectors of an angle of a triangle trisected the opposite side, these lines would be *both* perpendicular to that side.

[Mr. McCormick remarks that the proof may be effected by the aid of the *first* book of Euclid (as well as by Euc. vi., 3), and refers to Lardner's *Euclid*, Book i., Prop. 26, Cor. 1.]

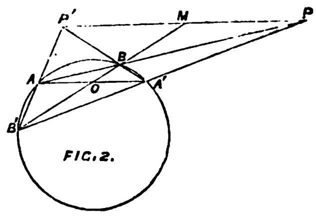
1881. (Proposed by J. R. Allen.)—Let O be the middle point of any chord AA′ of a circle ABA′; through O draw any other chord BOB′; join AB, A′B′ and produce these lines to meet in P. Show that the locus of P is a straight line parallel to AA′; and is the same as that of the intersection of tangents drawn to the circle at B, B′.

Solution by J. H. Taylor, B.A.; A. Renshaw; *and others.*



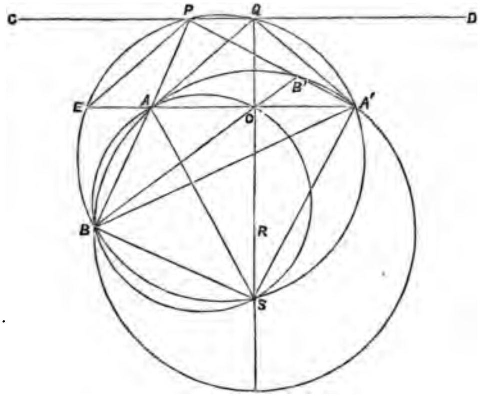
Let R and R′ (Fig 1) be the intersections of pairs of tangents at A, A′ and B, B′. Join RR′.

Then $CO \cdot CR = CA^2 = CB'^2 = CO' \cdot CR'$, therefore the triangles CO′O, CRR′ are similar, for they have a common angle and the sides about it proportional; therefore $\angle CRR' = CO'O =$ a right angle, therefore RR′ is the polar of O.



Next (Fig 2) let AB, A′B′ intersect in P, and AB′, A′B in P′; then, by a property of a complete quadrilateral, (P′ . B′OA′P) is an harmonic pencil, and AO = OA′; therefore the transversal AA′ is parallel to P′P. Let B′B meet P′P in M; then (B′OBM) is an harmonic range; therefore M is on the polar of O: but P′P is parallel to AA′, therefore P′P is the polar of O: and it is also the locus of the intersection of tangents at B and B′, by the first part.

II. *Solution by the* PROPOSER; H. MURPHY; E. McCORMICK; *and others.*



Through O draw the diameter ROQ; through P draw CD parallel to AA′, cutting the diameter ROQ in Q; round the points B, A, O draw a circle cutting the diameter ROQ in S; and join SA, SA′, SB.

Now, ∵ ∠AOS = a right ∠ (Euc. iii. 3),

∴ ∠ABS = a right ∠ (Euc. iii. 22) = OBA + OAS = OA′B′ + OA′S;

∴ the whole ∠ PA′S = ABS = a right ∠.

Also, ∵ CD is parallel to AA′ and SQ cuts AA′ at right ∠s,

∴ ∠PQS = a right ∠ (Euc. i. 29),

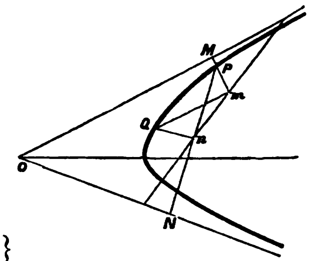
therefore a circle may be drawn round the points P, Q, A′, S, B. Produce A′A to meet this circle in E, and join PE, QA, QA′, BA′.

Now, since PQ is parallel to EA′, it may easily be proved that PE = QA′ = QA, therefore QA is parallel to PE, therefore ∠QAA′ = PEA′ = PBA′, therefore QA is a tangent to the circle BAA′; hence the point Q and the line CD are given in position. Also it is well known that the locus of Q is the same as that of the intersections of tangents at B, B′; it may also be proved geometrically by drawing circles round QARA′ and BRB′.

1899. (Proposed by G. O. HANLON.)—If perpendiculars from any point P on a hyperbola are drawn to the asymptotes, prove that the line joining the feet of perpendiculars from any other point Q on the curve to these lines passes through a fixed point F. Also find the locus of F, as P varies on the curve, and determine the hyperbola for which this locus reduces to a point.

I. *Solution by* J. H. TAYLOR, B.A.; *the* PROPOSER; A. RENSHAW; *and others*.

Taking the asymptotes as axes, and putting ω for the angle of ordination MON, let the equation of the hyperbola be $xy = k^2$; and let (x', y'), and (x'', y'') be the coordinates of the points P and Q. Then the respective equations of the lines PN, Qn, PM, Qm, are



$$y-y' = (x'-x) \sec \omega, \; y = y'',$$
$$x-x' = (y'-y) \sec \omega, \quad x = x'',$$

Hence the equation of mn is

$$k^2 \left\{ yy' + xy' \cos \omega - \cos \omega \, (k^2 + y'^2 \cos \omega) \right\}$$
$$- y'' \, (yy'^2 \cos \omega + xy'^2 - y'^3 \cos \omega - k^2 y' \cos^2 \omega) = 0,$$

which contains the indeterminate y'' and therefore passes through the point given by

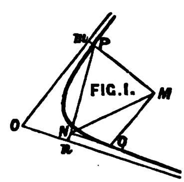
$$yy' + xy' \cos \omega - \cos \omega \, (k^2 + y'^2 \cos \omega) = 0,$$
$$yy' \cos \omega + xy' - \cos \omega \, (y'^2 + k^2 \cos \omega) = 0;$$

that is, through the fixed point (F) whose coordinates are $(y' \cos \omega, \; x' \cos \omega)$. The locus of F when (x', y') varies is at once seen to be the hyperbola $xy = k^2 \cos^2 \omega$, which reduces to the origin if $\cos \omega = 0$, that is if the hyperbola is rectangular.

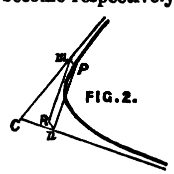
II. *Solution by* W. H. LAVERTY.

Let us represent by [X] the anharmonic ratio of any four points of a series of which X is one point, always supposing that if [X] = [Y], [Y] is the anharmonic ratio of four *corresponding* points of the second series.

Let P, Q (Fig. 1) be the two points on the conic; Pm, Pn; QM, QN; the perpendiculars from P and Q, on the asymptotes, and on Pm, Pn, respectively. Then $\angle$ NPM = mCn; and if P be fixed, we have [M] = [N]; moreover, these homographic series are in perspective, therefore MN envelopes a point, *i.e.* always passes through a fixed point.



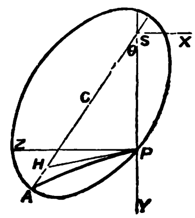
Next, having shown that MN always passes through a fixed point, we may, to solve the latter part of the problem, take any two points Q, Q, keeping which points fixed, we find the locus of the intersection of MN and M'N' as P varies. Take then the points at infinity on the asymptotes. Then MN, M'N', become respectively the lines through m, n, parallel to Pn, Pm, (Fig. 2); let their intersection be R.



Now, P[m] = P[n], therefore [m] = [n]; and these homographic series are not in perspective, therefore mn envelopes a conic. Again, in the triangle Rmn; m and n always lie on fixed lines; Rm, Rn, (being perpendicular to the asymptotes) always pass through fixed points at infinity, and mn envelopes a conic, therefore the locus of R is a conic, which evidently degenerates into a point (the origin) when the hyperbola is equilateral.

1722. (Proposed by R. TUCKER, M.A.)—A perfectly elastic ball is dropped from the fixed focus of a perfectly smooth ellipse; supposing the position of the ellipse to vary, find the locus of the vertices of the curves described after impact on the elliptic arc.

Solution by the PROPOSER.



Take one position of the ellipse, of which S, H are the foci and C, A the centre and vertex respectively. Let the chord of descent SP ($=r$) make the $\angle$ ASP $=\theta$ with the major axis, and through P draw the horizontal line PZ: then since the ball is perfectly elastic, it may, after impact, be considered as projected in the direction PH with the velocity acquired in SP; hence, putting $\epsilon = \angle$ HPZ, the equation to its path, referred to horizontal and vertical axes SX, SY through S is

$$r-y = x\tan\epsilon - \frac{x^2}{4r\cos^2\epsilon}.$$

The coordinates of the vertex are

$$X = r\sin 2\epsilon,\quad Y = r\cos^2\epsilon,$$

whence, combining with the equations of condition

$$2a-r = 2ae\sin\theta\sec\epsilon,\quad c = r(1-e\cos\theta),$$

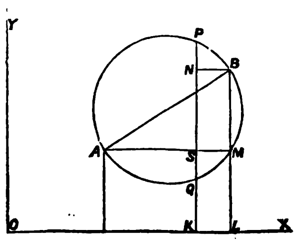
we obtain for the locus required

$$(X^2+4Y^2)^3-16a(X^2+4Y^2)^2\left\{Y+ae^2+a\right\}+64a^2Y(X^2+4Y^2)(Y+2c)$$
$$+256a^2c^2Y^2 = 0.$$

1963. (Proposed by M. W. CROFTON, B.A.)—Show that the equation of a circle on the line joining (x', y') and (x'', y'') as diameter is

$$(x-x')(x-x'')+(y-y')(y-y'')=0.$$

Solutions by T. J. SANDERSON, B.A.; H. TOMLINSON; REV. R. H. WRIGHT, M.A.; J. DALE; R. TUCKER, M.A.; S. W. BROMFIELD; A. COHEN, B.A.; REV. J. L. KITCHIN, M.A.; W. H. LAVERTY; *the* PROPOSER; *and many others.*



1. Let A, B, be the points (x', y'), (x'', y'') respectively; P, or (x, y), any point on the circle described on AB as diameter. Draw the ordinate PK meeting the circle again in Q, and the ordinate BL meeting the circle again in M. Join AM, which will be perpendicular to BL, since AB is a diameter. Let AM meet PK in S. Then, if BN be perpendicular to PQ, we have AS . SM = PS . SQ = PS . PN (if BN be perpendicular to PQ),

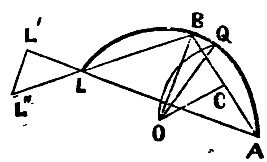
or $$(x-x')(x''-x) = (y-y')(y-y''),$$

that is $(x-x')(x-x'')+(y-y')(y-y'') = 0$, the equation required.

2. *Otherwise:* The given equation manifestly represents a circle, since the coefficients of the highest powers are the same, and this circle passes through the points (x', y'), (x'', y''), (x', y''), (x'', y'); and these are the angular points of a rectangle, hence the equation represents a circle on the diagonal formed by joining (x', y') with (x'', y'') as diameter.

1759. (Proposed by A. RENSHAW.)—Travelling at night on a line of railway of circular form, I noticed three distant lights L, L′, L″. At a certain station A, the lights L and L′ appeared to coalesce; and at the next station B, the lights L and L″ assumed the same appearance; the distances LL′, LL″, L′L″ are k, l, m, respectively, the direct distance from A to B is n, and the circle of which the Railway forms a part would if completed pass through L. Now if with radius of this circle and centre A, another circle were described, cutting the Railway between A and B in Q, find the distance along the line from Q to B.

Solution by E. L. ROSOLIS; *the* PROPOSER; *and others.*



From O, the centre of the circle ABL, draw OC perpendicular to AB; then, putting $\angle$ L′LL″ = ALB = BOC = α°, this angle is given by the relation $2kl \cos\alpha = k^2+l^2-m^2$; moreover OB $= \frac{1}{2}n \operatorname{cosec}\alpha$; and since the arc AQ subtends an angle of 60° at O, we have $\angle$ BOQ $= 2\alpha^\circ - 60^\circ$; and consequently arc BQ $= \dfrac{\alpha^\circ - 30^\circ}{180}\pi n \operatorname{cosec}\alpha$.

1900. (Proposed by A. RENSHAW.)—The Austrian Government have lately issued a loan of 734,694 Bonds of £19 17*s*. 0*d*. sterling, or 500 francs, or 200 florins Austrian, value in silver; and it appears that a contract for it has been entered into between the Imperial Government of Austria and the Comptoir d'Escompte of Paris, in combination with several capitalists. The Bonds will be issued at £13. 14*s*. 6*d*. each, with coupons attached, payable half-yearly, of the value of 9*s*. 11*d*. each, being at the rate of 5 per cent. per annum on the par value of £19. 17*s*. 0*d*. from the 1st December, 1865. They will be redeemed in 37 years by half-yearly drawings, to take place publicly, at the Austrian Embassy in Paris, on the 1st May and 1st November of each year. At each drawing an equal number of Bonds, viz. 9,928, will be withdrawn and paid of at par (£19. 17*s*. 0*d*.) with the half-yearly dividend. Find, from these data, the rate of interest at which the Austrian Government are thus borrowing.

Solution by SAMUEL BILLS.

In the first place, the sum actually received by the Austrian Government on account of the loan will be

$$£13.\ 14s.\ 6d. \times 734694 = £10,083,675.\ 3s.\ 0d. \ldots\ldots\ldots (A).$$

The amount paid back to the Bondholders over and above the principal, will be composed of two parts. The first part will be the difference between £19. 17s. 0d. and £13. 14s 6d. multiplied by 734694, that is,

$$£6.\ 2s.\ 6d. \times 734694 = £4,499,820.\ 15s.\ 0d. \ \ldots\ldots\ldots (B).$$

The second part will be composed of the amount of the coupons paid at the several half-yearly drawings. Put $734694 = a$, and $9928 = b$; then the several amounts paid at the 1st, 2nd, 3rd,.... 73rd, 74th drawings will be

$$a\,(9s.\ 11d.),\ (a-b)\,(9s.\ 11d.),\ (a-2b)\,(9s.\ 11d.) \ldots (a-72b)\,(9s.\ 11d.),$$
$$9950\,(9s.\ 11d.).$$

The truth of the last will appear by considering that

$$73 \times 9928 + 9950 = 734694.$$

Now, the sum of the first 73 terms of the preceding series is

$$73\,(a-36b)\,(9s.\ 11d.) = £13,656,181.\ 3s.\ 6d.$$

Also, 9950 (9s. 11*d.*) = £4933. 10s. 10d. Adding these together, we have, as the total paid on account of the coupons,

$$£13,661,114.\ 14s.\ 4d. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (C).$$

Adding together (B) and (C), we obtain, as the total amount paid by the Austrian Government to the Bondholders, over and above the amount of the principal,

$$£18,160,935.\ 9s.\ 4d. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (D).$$

Let x denote the rate per £. per half year; then, putting £13. 14s. 6d. $=p$, we shall have the respective amounts of interest corresponding to the 1st, 2nd, 3d,......73rd, last half-years,

$$apx,\ \ (a-b)\,px,\ \ (a-2b)\,px,\ \ldots\ldots(a-72b)\,px,\ \ 9950px.$$

Hence, by addition, we should have

$$73\,(a-36b)\,px + 9950px - (D).$$

Restoring the values of a, b, p and (D), we should find £9. 12s. $1\frac{1}{4}$d. nearly as the rate per cent. per annum required.

OBSERVATIONS ON THE "THREE AND FOUR POINT PROBLEMS" IN THEIR RELATIONS TO INFINITY. BY W. S. B. WOOLHOUSE, F.R.A.S.

The three point problem, first proposed by me in the *Lady's and Gentleman's Diary* for 1861, and reproduced in the *Educational Times* for December, 1862, as Question 1333, was enunciated thus:—

"Three points being taken at random in space as the corners of a plane triangle determine the probability that it shall be acute."

In the Solutions that have been given to this problem, the three points are first conceived to be limited to the volume of a given sphere; and as the

resulting value of the probability, viz., $\frac{33}{70}$, is obviously independent of the magnitude of the sphere, it is assumed to apply to the extreme case in which the radius is supposed to be infinite.* According to this process, unlimited space is represented by a sphere of infinite radius, to which conformation the mind would seem to be naturally led by an idea of symmetry in all directions round a finite region. Other reasons of a practical nature might be adduced in support of the general consistency of this hypothesis, with reference to the subject under consideration, giving to it, in some respects at least, a preference to what might otherwise be regarded as more strictly legitimate in the abstract. My mind is fully made up on this point, and I may be induced to discuss it hereafter. There is one thing quite certain, that whatever hypothesis is taken ought to be consistent with itself. Also, to comprehend every combination of the associated points, it is an essential condition that they shall severally and alike occupy every assignable position.

The present communication is for the purpose of briefly adverting to the untenable character of certain statements relative to this problem, contained in an interesting paper by Professor DE MORGAN, "On Infinity, &c.," printed in the *Transactions of the Cambridge Philosophical Society*, Vol. XI., Part I., page 3. It is probable that Professor DE MORGAN may have become aware of the logical deficiency of the reasoning advanced by him, if he has since given the subject any attention. Under any circumstances, I am assured that an earnest endeavour to throw some additional light on a subject, the difficulties of which are so critical and so peculiarly calculated to create differences of opinion, will be received with his characteristic liberality and independence of thought. To avoid, if possible, the risk of being obscure, I shall first transcribe, *in extenso*, the statements in question, which are the following:—

"The absolute infinite is avoided by recourse to increase without limit. If, for instance, we have to choose points in space which shall satisfy certain conditions, and if we first choose them within a given sphere, and then increase the radius of the sphere without limit, do we not finally allow unlimited choice? What point of space is omitted out of a sphere of infinite radius? Certainly not any assignable point. But, on the other hand, we know well that we dare not deny of an infinite sphere anything which is true of *any sphere however great*: if there be anything which is true, and is equally true of all the increasing spheres, that truth is, with obvious reason and invariable success, predicated of the sphere increased *ad infinitum*. Now it is certainly true of any sphere however great, that there is infinitely more space outside than inside. If this be also true of an infinite sphere, that sphere does not include all space: if it be false, where does the sphere begin to include what it does not include? Where does the ever remaining external space lose that character? Let us see which of the two assertions will a problem justify."

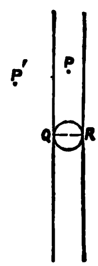
"Three points are taken at hazard in space, all points being equally likely: what is the chance of the triangle having three acute angles? I saw this problem solved in a mathematical journal, by a sound use of the integral calculus, on the supposition of the points being taken within a given sphere. The answer gave a finite chance for an acute-angled triangle, which remained finite when the sphere was enlarged *ad infinitum*. But it is very easily shown that the chance of an acute-angled triangle must be infinitely small. Take any one base; at its ends, draw two planes perpendicular to it, and infinitely extended: also, on the base as a diameter, describe a sphere. An

* The *Reprint*, Vol. IV., page 80, contains a note from me, respecting an elementary property, that the mathematical value of the probability is unaltered when one of the points is restricted to a fixed position on the surface of the sphere.

acute-angled triangle must have its vertex within the infinite strip between the parallel planes, and outside the sphere. Now it is clear that the possible vertices outside the strip infinitely outnumber those within the strip, if points be equally distributed through space. For any given base, and consequently for any number of bases, there is no appreciable chance of an acute-angled triangle: the same then for all bases and unlimited choice, if any one base be as likely as any other."

The following objections arise upon these statements. In the first place, it may be asserted that a sphere of infinite radius can have no existing boundary, if we accept the term infinite in an absolute sense; which we must do here, as the sphere is specifically adopted as the representative of infinite space, and does not admit any consideration of different orders of infinity. An infinitely distant boundary is equivalent to no boundary at all, since any conception whatever of boundary would instantly negative the idea of an infinite sphere. Again, when a sphere is taken and accepted as the representative of infinite space, it is evident that an inconsistency amounting to no less than a direct contradiction, is immediately introduced by any assumption of the existence of points outside it.

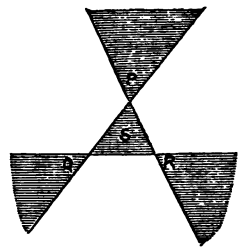
In the second paragraph, it is professed to be shown, that the chance of an acute triangle is infinitely small instead of a determinate finite value. The fallacy, however, is introduced by taking an assumed base QR, and always treating it as finite, and at the same time allowing the vertex P to roam uncontrolled over infinite space, both within and without the strip in the manner described. Such constructions are necessarily partial, defective, and therefore erroneous, since the points Q, R should be alike privileged to traverse every region of infinite space to which the point P has been admitted. As the case stands, they are mutually located within distances which, however great, are exclusively supposed to be finite. That is, to put the matter briefly, Professor DE MORGAN, in a most exceptional manner, assigns a range to the third point infinitely greater than that which is virtually accorded by him to the points Q and R. These defects necessarily lead to wrong conclusions.



Professor DE MORGAN next advances certain views with respect to the "four point problem," and arrives at an even chance between a convex and a re-entering quadrilateral. The solution given by him is as follows:—

"This question about the distinction between space with infinitely distant boundary and infinite space, will always raise discussion. Any triangle, *however great*,—and therefore an infinite triangle,—has six external spaces, each infinitely greater than the triangle. Three, say P, Q, R, are the angular spaces of the opposite angles; and, S being the triangle, the other three are angular spaces with the triangle cut off, or $P-S$, $Q-S$, $R-S$. If four points be taken quite at hazard, all points of the plane being equally likely, all triangles are equally likely to be formed by three of the points. Any one triangle being taken, the chance that the fourth point so falls that one of the four shall be in the triangle of the other three is to the chance against it as $P+Q+R+S$ to $P+Q+R-3S$; that is, S being infinitely small compared with P, or Q, or R, the chance is an even chance. Hence, by repeating all the equally probable triangles on common principles, it is an even chance, four points being taken at hazard, that one shall be in the triangle of the other three. I am aware of a very ingenious proposal of solution which gives 1 to 3 instead of 1 to 1, and I am prepared to discuss it when it shall be published. I will not at present pronounce decidedly for the above solution; but I am myself utterly unable to see how it can be questioned."

It is here obviously true of a *finite* triangle, *however great,* that it must have the six external spaces as exhibited in the annexed diagram. But again, exception must be taken to the validity of the proposition which asserts that an *absolutely infinite* triangle can have such adjuncts. Any conception of such external spaces, or indeed of the sides of the triangle, must at once put a direct negative upon its presumed infinite character, which is tantamount to a positive contradiction.



In order that the aggregate combinations may be complete, it is clear that all the four points should alike pass through every possible change of position, and therefore that the vertices or corners of the assumed triangle should severally and jointly transverse every region of space, which the fourth point has been supposed to occupy. It is manifestly wrong, therefore, to suppose that the triangle S is in all cases infinitely small compared with P, Q, or R; and the probability deduced from this supposition must be erroneous. Professor De Morgan is instinctively right in his hesitation to pronounce decidedly in favour of this method.

The substance of a true solution for the general case in which the four points are supposed to be taken at random within any given enclosed area may be sketched thus. Let P, Q, R be three of the points; then, as the fourth point S may be posited anywhere within the proposed area, it is evident that the probability that it shall fall within the triangle PQR is determined by the fraction $\frac{\text{triangle PQR}}{\text{given area}}$. Therefore, when the points occupy all positions on the given area, we shall have

$$\begin{aligned}\frac{\text{average triangle}}{\text{given area}} &= \text{prob. of S within PQR}\\ &= \quad ,, \quad \text{P} \quad ,, \quad \text{QRS}\\ &= \quad ,, \quad \text{Q} \quad ,, \quad \text{RSP}\\ &= \quad ,, \quad \text{R} \quad ,, \quad \text{SPQ}.\end{aligned}$$

These four separate probabilities are by symmetry precisely identical in value, since the positions of the four points admit of being mutually interchanged or permuted. The several conditions are also individually exclusive, and the sum of the four equal values obviously makes up the complete probability that the quadrilateral shall be re-entering. This last probability is therefore $\frac{\text{4 times average triangle}}{\text{given area}}$; that is, the required probability of a re-entering quadrilateral is found by comparing four times the average area of all inserted triangles with the given area.

If the proposed area be triangular, Mr. Stephen Watson and Professor Sylvester have shown (Quest. 1229, *Reprint,* Vol. II. p. 95, and Vol. IV. p. 101) that the average inserted triangle is $\frac{1}{12}$th of the given area. Hence, when the four points are to be taken on the surface of any given triangle, the probability of a re-entering quadrilateral is $\frac{1}{3}$.

1503. (Proposed by Professor SYLVESTER.)—A table has n holes bored in its rim, into which ν pegs are to be inserted at random, ν being not greater than $\frac{1}{2}n$. Show that the probability of there being no two pegs without one or more unoccupied holes between them will be equal to $\frac{\pi(n-\nu).\pi(n-\nu-1)}{\pi(n-1).\pi(n-2\nu)}$, and, if ν is given, approaches to certainty as n becomes indefinitely great.

[N.B.—$\pi(n)$ here denotes the product $1.2.3....n$.]

I. *Solution by* SAMUEL ROBERTS, M.A.

We may suppose the same peg to make the circuit of the table, both when we seek the possible and when we seek the favourable cases. Hence the ratio of these numbers will be the same as that of the corresponding numbers when one hole is occupied by a peg.

On this supposition the number of possible cases is the number of variations of $n-1$ things taken $\nu-1$ together, or $\frac{\pi(n-1)}{\pi(n-\nu)}$. Let a circle be drawn with $n-\nu$ radii which represent the $n-\nu$ holes, which necessarily remain unoccupied. We may consider these radii as fixed, while the $n-\nu$ spaces between them are possible favourable positions of the pegs. One of these spaces being occupied, the corresponding number of favourable cases is the number of variations of $n-\nu-1$ things taken $\nu-1$ together, or $\frac{\pi(n-\nu-1)}{\pi(n-2\nu)}$, and the ratio of these values is $\frac{\pi(n-\nu)}{\pi(n-1)}.\frac{\pi(n-\nu-1)}{\pi(n-2\nu)}$.

When the expression is put in the form of a fraction, having its numerator and denominator of the same degree in n, we see that as n becomes great, the value approaches unity, if ν remains constant.

II. *Solution by* G. C. DE MORGAN, M.A.

Let $z_{n,\nu}$ be the number of ways in which ν pegs may be inserted in n holes, in accordance with the conditions of the problem, on the supposition that the holes are bored in a row of which the first and last are not to be considered as adjacent. To get the number of ways in which they may be inserted when the holes are bored completely round the table, we must subtract the number of ways in which the first and last are both filled up, or $z_{n-4,\nu-2}$.

Now $z_{n,\nu}$ is made up of the number of ways in which the first hole is filled, together with the number of ways in which it is not filled; hence

$$z_{n,\nu} = z_{n-2,\nu-1} + z_{n-1,\nu}.$$

Let $z_{\nu,n}$ be the coefficient of $x^n y^\nu$ in $\phi(x, y)$: then we have

$$\phi(x,y) = z_{0,0} + z_{1,0} \cdot x + z_{2,0} \cdot x^2 + \ldots + z_{1,1} \cdot xy + z_{2,1} \cdot x^2y + \ldots$$
$$x \cdot \phi(x,y) = z_{0,0} \cdot x + z_{1,0} \cdot x^2 + \ldots\ldots\ldots\ldots + z_{1,1} \cdot x^2y + \ldots$$
$$x^2y \cdot \phi(x,y) = z_{0,0} \cdot x^2y + z_{1,0} \cdot x^3y + \ldots\ldots + z_{1,1} \cdot x^3y^2 + \ldots$$

From these and the equation of differences above, remembering that $z_{0,0} = 1$, $z_{1,0}=1$, &c., $z_{1,1}=1$, we get $\left\{1-x-x^2y\right\} \cdot \phi(x,y) = 1+xy$, therefore $\phi(x,y) = \dfrac{1+xy}{1-x-x^2y} = 1+xy+x\cdot(1+xy)^2+x^2\cdot(1+xy)^3+\ldots$

The term obtained from $x^{k-1} \cdot (1+xy)^k$ which has y^ν as a factor, is $\dfrac{k\cdot(k-1)..(k-\nu+1)}{1\cdot2\ldots\nu} \cdot x^{\nu+k-1} \cdot y^\nu$, and we must take k in such a manner that $\nu+k-1 = n$, or $k = n-\nu+1$. This gives for the coefficient of x^ny^ν, $\dfrac{(n-\nu+1)\cdot(n-\nu)\ldots(n-2\nu+2)}{1\cdot2..\nu}$, or, using Professor SYLVESTER'S notation, $\dfrac{\pi\cdot(n-\nu+1)}{\pi(\nu)\cdot\pi(n-2\nu+1)} = z_{n,\nu}$.

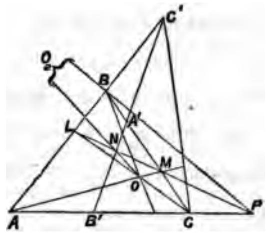
$$\text{Hence } z_{n,\nu} - z_{n-4,\nu-2} = \frac{\pi(n-\nu+1)}{\pi(\nu)\cdot\pi(n-2\nu+1)} - \frac{\pi(n-\nu-1)}{\pi(\nu-2)\cdot\pi(n-2\nu+1)}$$
$$= n\cdot\frac{\pi\cdot(n-\nu-1)}{\pi(\nu)\cdot\pi(n-2\nu)}.$$

Dividing by $\dfrac{\pi(n)}{\pi(\nu)\cdot\pi(n-\nu)}$, the total number of ways in which ν pegs may be inserted in n holes, we get $\dfrac{\pi(n-\nu)\cdot\pi(n-\nu-1)}{\pi(n-1)\cdot\pi(n-2\nu)}$, the probability required. The limit of this, when n is infinite, is given by substituting $\sqrt{(2\pi)}\cdot(n-\nu)^{n-\nu+\frac{1}{2}}\cdot\epsilon^{-(n-\nu)}$ for $\pi(n-\nu)$, &c. This gives, after reduction,

$$\frac{\left(1-\frac{\nu}{n}\right)^{n-\nu+\frac{1}{2}}\cdot\left(1-\frac{\nu+1}{n}\right)^{n-\nu-\frac{1}{2}}}{\left(1-\frac{1}{n}\right)^{n-\frac{1}{2}}\cdot\left(1-\frac{2\nu}{n}\right)^{n-2\nu+\frac{1}{2}}},$$ the limit of which for a given value of ν is $\dfrac{\epsilon^{-\nu}\cdot\epsilon^{-(\nu+1)}}{\epsilon^{-1}\cdot\epsilon^{-2\nu}}$, or 1.

1799. (Proposed by Professor WHITWORTH, M.A.)—Three conics are described so that each of them passes through the same point O, and through the extremities of two of the diagonals of the same complete quadrilateral. Prove that if O_1, O_2, O_3 are their other points of intersection, then OO_1, OO_2, OO_3 are the tangents to the three conics at O.

I. *Solution by* J. DALE; H. TOMLINSON; *and others.*



Let AA′, BB′, CC′ be the diagonals of the quadrilateral; A′, B′, C′ being collinear vertices; and O the common point. Draw the lines AOM, cutting BC in M; COL, cutting AB in L; and LNMP, cutting B′C′ in N and AC in P; also join ON, PB, and produce these lines to meet in O_2. In the hexagon $AA'BB'OO_2$ the lines OA, BA′; AB′, O_2B; B′A′, O_2O intersect in the collinear points M, P, N; therefore, by Pascal's theorem, O_2 lies on the conic AA′BB′O. In the hexagon $BB'CC'OO_2$ the lines OC, C′B; CB′, BO_2; B′C′, O_2O; intersect in the collinear points L, P, M; therefore O_2 lies on the conic BB′CC′O. In the pentagon CC′AA′O, the straight line OO_2 is drawn through O so that the lines OC, C′A; CA′, AO; A′C′, OO_2 intersect in the collinear points L, M, N; therefore OO_2 is a tangent to the conic CC′AA′O. In the other two cases the proof is similar.

II. *Solution by the* PROPOSER; W. H. LAVERTY; E. MCCORMICK; W. CHADWICK; *and others.*

Let AA′, BB′, CC′ be the diagonals of the quadrilateral; and, taking the four sides BC, CA, AB, A′B′C′ as lines of reference for quadrilinear coordinates, let $(\alpha', \beta', \gamma', \delta')$ be the coordinates of O. Then we may write the equations to the conics which pass through $OO_2O_3BB'CC'$, $OO_3O_1CC'AA'$, $OO_1O_2AA'BB'$, respectively

$$S_1 \equiv \frac{\alpha\delta}{\alpha'\delta'} - \frac{\beta\gamma}{\beta'\gamma'} = 0,\quad S_2 \equiv \frac{\alpha\beta}{\alpha'\beta'} - \frac{\gamma\delta}{\gamma'\delta'} = 0,\quad S_3 \equiv \frac{\alpha\gamma}{\alpha'\gamma'} - \frac{\beta\delta}{\beta'\delta'} = 0.$$

Now $S_2 - S_3 \equiv \frac{\alpha}{\alpha'}\left(\frac{\beta}{\beta'} - \frac{\gamma}{\gamma'}\right) - \frac{\delta}{\delta'}\left(\frac{\gamma}{\gamma'} - \frac{\beta}{\beta'}\right) \equiv \left(\frac{\alpha}{\alpha'} + \frac{\delta}{\delta'}\right)\left(\frac{\beta}{\beta'} - \frac{\gamma}{\gamma'}\right)$;

hence $\frac{\alpha}{\alpha'} + \frac{\delta}{\delta'} = 0$ and $\frac{\beta}{\beta'} - \frac{\gamma}{\gamma'} = 0$ are the equations to a pair of common chords of S_2 and S_3. But $\frac{\beta}{\beta'} - \frac{\gamma}{\gamma'} = 0$ is the equation to the chord OA; therefore $\frac{\alpha}{\alpha'} + \frac{\delta}{\delta'} = 0$ must represent the chord O_1A'.

Similarly we can show that

$$\frac{\beta}{\beta'} + \frac{\gamma}{\gamma'} = 0 \text{ must represent the chord } O_1A.$$

Hence the point O_1 is given by the equations

$\frac{\alpha}{\alpha'} + \frac{\delta}{\delta'} = 0$ and $\frac{\beta}{\beta'} + \frac{\gamma}{\gamma'} = 0$; and it therefore lies upon the line whose equation is

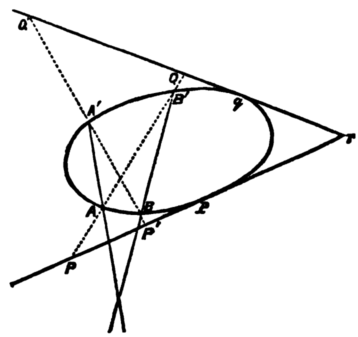
$$\frac{\alpha}{\alpha'} + \frac{\delta}{\delta'} - \left(\frac{\beta}{\beta'} + \frac{\gamma}{\gamma'}\right) = 0,\ \text{ or } \frac{\alpha}{\alpha'} - \frac{\beta}{\beta'} - \frac{\gamma}{\gamma'} + \frac{\delta}{\delta'} = 0.$$

But this is known to be the equation to the tangent at $(\alpha', \beta', \gamma', \delta')$, or O, to the conic S_1; that is to say, OO_1 is the tangent at O to S_1.

Similarly OO_2 and OO_3 are the tangents at O to the other conics.

1941. (Proposed by F. D. THOMSON, M.A.)—**AA′B′B is a quadrilateral** inscribed in a conic. Two tangents PP′, QQ′ meet the **diagonals AB′, A′B** in the points P, P′, Q, Q′ respectively. Show that a conic can be described so as to touch AA′, BB′, and also pass through the four points P, P′, Q, Q′.

Solution by the PROPOSER.



Let $LM = R^2$ be the equation to the conic, referred to the two tangents and their chord of contact. Let a, b, a', b', be the quantities defining as usual the points A, B, A′, B′. Then the equation to a conic round PP′QQ′ is of the form

$[AB'][A'B] = \lambda[PP'][QQ']$ (i).

Now the equations to AB′, A′B are, respectively,

$ab'L-(a+b')R+M=0\,..\,[AB'],$

$a'bL-(a'+b)R+M=0\,..\,[A'B];$

therefore (i.) becomes

$$\{ab'L-(a+b')R+M\}\{a'bL-(a'+b)R+M\} = \lambda LM \;\ldots\ldots\ldots\;\text{(ii)}.$$

The equation to AA′ is $aa'L-(a+a')R+M = 0$ .... (iii); therefore to find when (iii.) meets (ii.), we have

$$\{(ab'-aa')L+(a'-b')R\}\{(a'b-aa')L+(a-b)R\}$$
$$= \lambda L\{(a+a')R-aa'L\},$$

or, $$(b'-a')(b-a)(aL-R)(a'L-R) = \lambda L\{(a+a')R-aa'L\}$$

or, $$aa'L^2\{(b'-a')(b-a)+\lambda\}-LR(a+a')\{(b'-a')(b-a)+\lambda\}$$
$$+(b'-a')(b-a)R^2 = 0\ldots\ldots\ldots\text{(iv.)},$$

therefore if AA′ touch (ii), we must have

$$(a+a')^2\{(b'-a')(b-a)+\lambda\}^2 = 4aa'\{(b'-a')(b-a)+\lambda\}(b'-a')(b-a),$$

therefore $$(b'-a')(b-a)+\lambda = 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v.)}$$

or, $$(a+a')\{(b'-a')(b-a)+\lambda\} = 4aa'(b'-a')(b-a).$$

The value of λ given by (v.) is unchanged if we write b, b' for a, a', and therefore if (ii.) touches AA′ it also touches BB′.

COR. 1.—If we substitute for λ in (ii.) from (v.) the equation to the conic becomes

$$\{ab'L-(a+b')R+M\}\{a'bL-(a'+b)R+M\}+(b'-a')(b-a)LM = 0,$$

or, $$aa'bb'L^2-\{ab'(a'+b)+a'b(a+b')\}LR+(aa'+bb')LM+M^2$$
$$-(a+a'+b+b')MR+(a+b')(a'+b)R^2 = 0,$$

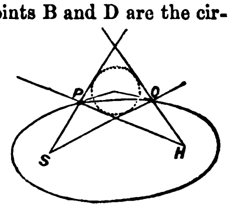
or, $$\{aa'L-(a+a')R+M\}\{bb'L-(b+b')R+M\}$$
$$+(a-b)(a'-b')R^2=0\ldots\ldots\ldots\ldots\ldots\ldots\text{(vi.)},$$

or, $$[AA'][BB']+(a-b)(a'-b')[pq]^2 = 0.$$

Hence pq is the chord of contact of the tangents AA′, BB′.

COR. 2.—Precisely the same work in *tangential* coordinates would lead to the reciprocal of this theorem, viz.: If ABCD be a quadrilateral circumscribing a conic, and P, Q any two points on the conic, then a conic can be described touching AP, AQ, CP, CQ and passing through the points B and D. Also the pole of PQ with respect to the first conic will be the pole of BD with respect to the second conic.

A particular case of the theorem is when the points B and D are the circular points at infinity, and therefore A and C the foci; and then we get the following theorem (noticed by Mr. C. TAYLOR in the *Messenger of Mathematics*):—that if P and Q be any points on a conic, S and H the foci, a *circle* can be inscribed in the quadrilateral formed by SP, SQ, HP, HQ, having its *centre* at the pole of PQ with respect to the conic.



1949. (Proposed by Professor CAYLEY.)—Find the conic of five-pointic intersection at any point of the cuspidal cubic $y^3 = x^2z$.

I. *Solution by* W. H. H. HUDSON, M.A.

Let P be the proposed point. Since a conic and a cubic intersect in 6 points, let Q be the other point of intersection. Let the tangent at P meet the cubic again in p, and let the tangent at p meet the cubic again in p', then shall PQp' be a straight line. For, if p'' be the point in which PQ meets the cubic again, the cubic made up of Pp taken twice and PQ intersect the given cubic in the 9 points $P^5 . p^2 . Q . p''$. Also the cubic made up of the conic and pp' meets the given cubic in the 9 points $P^5 . p^2 . Q . p'$; and 8 of these $P^5 . p^2 . Q$ being coincident, it follows, by a well-known theorem, that the 9th is so likewise, or p', p'' are the same point.

Let now $S=0$ be the equation of the conic; $u=0$, $v=0$, $w=0$ of the lines Pp, PQ, pp' respectively; then we must have

$$y^3 - x^2z = Sw - \lambda u^2 v.$$

We can form the equations $u=0$, $v=0$, $w=0$; then, equating coefficients, we have 10 equations to determine the 6 coefficients of $S=0$ and λ. Solving 7 of these equations, the values found satisfy the other three: and the result thus obtained is

$$5\left(\frac{x}{h}\right)^2 + 45\left(\frac{y}{k}\right)^2 - \left(\frac{z}{l}\right)^2 + 15\frac{yz}{kl} - 40\frac{zx}{lh} - 24\frac{xy}{hk} = 0,$$

h, k, l being the coordinates of the point of five-pointic intersection.

II. *Solution by the* PROPOSER.

The equation $y^3 = x^2z$, is satisfied by the values $x : y : z = 1 : \theta : \theta^3$; and conversely, to any given value of the parameter θ there corresponds a point

on the cubic $y^3 = x^2z$. Consider the five points corresponding to the values $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ respectively; the equation of the conic through these five points is

$$\begin{vmatrix} x^2, & y^2, & z^2, & yz, & zx, & xy \\ 1, & \theta_1^2, & \theta_1^6, & \theta_1^4, & \theta_1^3, & \theta_1 \\ \vdots & & & & & \end{vmatrix} = 0,$$

where the remaining four lines of the determinant are obtained from the second line by writing therein $\theta_2, \theta_3, \theta_4, \theta_5$ successively in place of θ_1. Writing for shortness $\zeta^{\frac{1}{2}}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ to denote the product of the differences of the quantities $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$, the equation contains the factor $\zeta^{\frac{1}{2}}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$, and we may therefore write it in the simplified form

$$\frac{1}{\zeta^{\frac{1}{2}}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)} \begin{vmatrix} x^2, & y^2, & z^2, & yz, & zx, & xy \\ 1, & \theta_1^2, & \theta_1^6, & \theta_1^4, & \theta_1^3, & \theta_1 \\ \vdots & & & & & \end{vmatrix} = 0.$$

Hence putting in this equation $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \phi$, we have the equation of the conic of five pointic intersection at the point (ϕ). The result in its reduced form may be obtained directly without much difficulty, but it is obtained most easily as follows: let the function on the left-hand of the foregoing equation be represented by

$$\left\{a, b, c, f, g, h\right\} (x, y, z)^{2}*$$

then writing $x : y : z = 1 : \theta : \theta^3$, we have

$$\begin{aligned}(a, b, c, f, g, h)\,(1, \theta, \theta^3)^2 &= \frac{1}{\zeta^{\frac{1}{2}}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)} \begin{vmatrix} 1, & \theta^2, & \theta^6, & \theta^4, & \theta^3, & \theta \\ 1, & \theta_1^2, & \theta_1^6, & \theta_1^4, & \theta_1^3, & \theta_1 \\ \vdots & & & & & \end{vmatrix} \\ &= \frac{(\theta-\theta_1)(\theta-\theta_2)(\theta-\theta_3)(\theta-\theta_4)(\theta-\theta_5)}{\zeta^{\frac{1}{2}}(\theta, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5)} \begin{vmatrix} 1, & \theta^2, & \theta^6, & \theta^4, & \theta^3, & \theta \\ 1, & \theta_1^2, & \theta_1^6, & \theta_1^4, & \theta_1^3, & \theta_1 \\ \vdots & & & & & \end{vmatrix} \\ &= (\theta-\theta_1)(\theta-\theta_2)(\theta-\theta_3)(\theta-\theta_4)(\theta-\theta_5)(\theta+\theta_1+\theta_2+\theta_3+\theta_4+\theta_5);\end{aligned}$$

[for the determinant, which is a function of the order 16 in the quantities $(\theta, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ conjointly, divides by $\zeta^{\frac{1}{2}}(\theta, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$, which is a function of the order 15; and as the quotient is a symmetrical function of $\theta\ \theta_1, \theta_2, \theta_3, \theta_4, \theta_5$, it must be equal, save to a numerical factor which may be disregarded, to $\theta+\theta_1+\theta_2+\theta_3+\theta_4+\theta_5$].

Hence if ϕ be the parameter of the given point, writing $\theta_1 = \theta_2 = \theta_3 = {}_4 = \theta_5 = \phi$, we have

$$\begin{aligned}\left\{a, b, c, f, g, h\right\} (1, \theta, \theta^3)^2 &= (\theta-\phi)^5\,(\theta+5\phi) \\ &= \left\{1, 0, -15, +40, -45, +24, -5\right\}(\theta, \phi)^6,\end{aligned}$$

where the left-hand side is

$$a + b\theta^2 + c\theta^6 + f\theta^4 + g\theta^3 + h\theta, = \left\{c, 0, f, g, b, h, a\right\} (\theta, 1)^6,$$

that is, we have

$$c = 1,\ f = -15\phi^2,\ g = 40\phi^3,\ b = -45\phi^4,\ h = 24\phi^5,\ a = -5\phi^6,$$

and the equation of the conic of five-pointic intersection therefore is

$$\left\{-5\phi^6, -45\phi^4, 1, -15\phi^2, 40\phi^3, 24\phi^5\right\} (x, y, z)^2 = 0,$$

* I write $\{\ \}$, instead of the usual arrow-headed parenthesis, to signify the omission of the binomial coefficients, viz., $\left\{a, b, c, f, g, h\right\}(x, y, z)^2$ means $ax^2+by^2+cz^2+fyz+gzx+hxy$.

or, what is the same thing,

$$-5\phi^6 x^2 - 45\phi^4 y^2 + z^2 - 15\phi^2 yz + 40\phi^3 zx + 24\phi^5 xy = 0,$$

which is the required result.

NOTE.—The condition in order that any six points $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$ of the cubic $y^3 = x^2 z$ may lie on a conic, is

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6 = 0.$$

1958. (Proposed by J. GRIFFITHS, M.A.)—Let P denote a point in the plane of a given triangle ABC; α, β, γ the feet of the perpendiculars drawn from P upon the sides BC, CA, AB. Then if the triangle $\alpha\beta\gamma$ is homologous with the given one ABC, the locus of P is a cubic which passes through (1) the angular points of the triangle; (2) the centres of the inscribed and escribed circles; (3) the point of intersection of the three perpendiculars; (4) the centre, O, of the circumscribing circle; (5) the points L, M, N, where the radii AO, BO, CO produced meet this circle again. Moreover, if P′ denote the inverse of P with respect to the sides of the given triangle, show that P′ also lies on the cubic-locus in question.

I. *Solution by* J. DALE; *and others.*

Let x, y, z be the coordinates of P, the equations of Aα, Bβ, Cγ will be

$$\frac{\beta}{y + x\cos C} = \frac{\gamma}{z + x\cos B}, \quad \frac{\gamma}{z + y\cos A} = \frac{\alpha}{x + y\cos C},$$

$$\frac{\alpha}{x + z\cos B} = \frac{\beta}{y + z\cos A};$$

and the condition that these three lines should meet in a point is

$$(y + z\cos A)(z + x\cos B)(x + y\cos C) = (y\cos A + z)(z\cos B + x)(x\cos C + y) \quad \text{(i.)},$$

or,
$$(\cos A - \cos B\cos C)\, x(y^2 - z^2) + (\cos B - \cos C\cos A)\, y(z^2 - x^2)$$
$$+ (\cos C - \cos A\cos B)\, z(x^2 - y^2) = 0 \quad \text{(ii.)}$$

(1.) This cubic passes through the angles of the triangle of reference, the coordinates of which are $(y = 0,\ z = 0)$, $(z = 0,\ x = 0)$, $(x = 0,\ y = 0)$.

(2.) It is satisfied by the values $x^2 = y^2 = z^2$, or

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1}, \quad \frac{x}{-1} = \frac{y}{1} = \frac{z}{1}, \quad \frac{x}{1} = \frac{y}{-1} = \frac{z}{1}, \quad \frac{x}{1} = \frac{y}{1} = \frac{z}{-1},$$

which represent the centres of the inscribed and escribed circles.

(3.) It is also satisfied by $x\cos A = y\cos B = z\cos C$, which represents the intersection of the perpendiculars.

(4.) The centre O of the circumscribing circle $\dfrac{x}{\cos A} = \dfrac{y}{\cos B} = \dfrac{z}{\cos C}$ lies on the cubic.

(5.) The points L, M, N, where the radii AO, BO, CO produced meet the circumscribing circle again, are given by the equations

$$\frac{x}{-\cos B\cos C}=\frac{y}{\cos B}=\frac{z}{\cos C},\quad \frac{x}{\cos A}=\frac{y}{-\cos C\cos A}=\frac{z}{\cos C},$$

$$\frac{x}{\cos A}=\frac{y}{\cos B}=\frac{z}{-\cos A\cos B};$$

and these values also satisfy the equation of the cubic.
for x, y, z; hence if any point P lie on the cubic, its inverse will also lie on

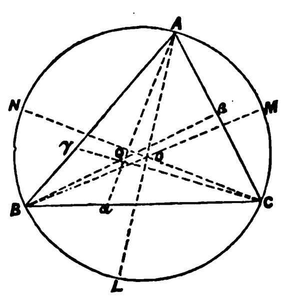
(6.) The equation remains unchanged when $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are substituted for
the cubic; (3) and (4) are examples of this.

(7.) The cubic is central, having O for its centre; (1) and (5) are examples of this.

[Mr. Greer remarks that the curve intersects each perpendicular of the triangle at infinity, three points which, *at first sight*, do not appear to satisfy the locus according to the geometrical condition by which it is defined. If the triangle ABC be turned round the centre of the circumscribing circle through two right angles, carrying the cubic with it, the curve returns into coincidence with its original position. Thus it is symmetrical all round this centre, which, therefore, must be one of its points of inflexion. Verifying this algebraically, there is brought to light the following identical relation, subsisting amongst the cosines of the angles of a plane triangle, viz., (writing A for cos A, &c.)

$$\begin{vmatrix} B^2-C^2, & C(B^2-A^2), & B(A^2-C^2) \\ C(B^2-A^2), & C^2-A^2, & A(C^2-B^2) \\ B(A^2-C^2), & A(C^2-B^2), & A^2-B^2 \end{vmatrix} = 0].$$

II. *Solution by* F. D. Thomson, M.A.



Let Q be the point $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ (trilinear coordinates). Draw AQα, BQβ, CQγ to meet the sides, and let the point Q be such that the perpendiculars at α, β, γ meet in a point P. We have to find the locus of P.

The equation to the perpendicular to BC through α must be of the form $kx+ny-mz=0$, where k is some constant, and since the line is perpendicular to $x=0$, we have $k+m\cos B-n\cos C=0$, therefore equation to perpendicular at α is

$$(n\cos C-m\cos B)\,x+ny-mz=0,$$

or, $\quad m(x\cos B+z)=n(x\cos C+y)$, or $m\,[BN]=n\,[CM]$......(iii.),

if L, M, N be the points on the circumscribing circle diametrically opposite to A, B, C, and [BN] denote the equation to BN.

Similarly equation to perpendicular at β is $n\,[CL]=l\,[AN]$ ......(iv.),

and equation to perpendicular at γ is $\quad l\,[AM]=m\,[BL]$......(v.);

therefore, eliminating l, m, n from (iii.), (iv.), (v.), we have

$$[BN]\,[CL]\,[AM]=[CM]\,[AN]\,[BL]\ \ldots\ldots\ldots\ldots(\text{vi.}),$$

the equation to a cubic through the intersections of BN with CM, AN, BL, that is to say, through ∞, N, B,

the equation to a cubic through the intersections of CL with CM, AN, BL, that is to say, through C, ∞, L,

the equation to a cubic through the intersections of AM with CM, AN, BL, that is to say, through M, A, ∞.

Therefore the cubic passes through A, B, C, L, M, N, and has its asymptotes perpendicular to the sides of the triangle.

Writing the equation in full, it will reduce to the form (ii.) of the first solution, which is seen to be satisfied by the points in (2), (3), (4); also by

$$\frac{x}{\cos A-\cos B\cos C}=\frac{y}{\cos B-\cos C\cos A}=\frac{z}{\cos C-\cos A\cos B}\text{.........(S)},$$

and the points where AS, BS, CS meet the sides BC, CA, AB.

It is seen from the form of the equation that if x, y, z satisfy it, $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ will also satisfy it. Hence the point

$$x(\cos A-\cos B\cos C)=y(\cos B-\cos C\cos A)=z(\cos C-\cos A\cos B)\ldots(S')$$

is also on the curve.

It may be shown that the tangents at A, B, C, S meet in S′, or with the definitions of a paper in the *Messenger of Mathematics*, (Vol. III., p. 15), that S′ is the *satellite* of each of the four parts A, B, C, S.

It may be shown, as in the paper referred to, that S is the satellite of each of the centres of the in- and e-scribed circles, and many similar properties may be deduced.

The above investigation suggests the question as to what is the locus of the point Q in the figure.

We have from the foregoing, if l, m, n be proportional to the coordinates of Q,

$$(n\cos C-m\cos B)\,x+ny-mz=0,$$

and two similar equations.

Hence, eliminating x, y, z, we get a cubic which reduces to

$$l\cos A\,(m^2\sin^2 B-n^2\sin^2 C)+m\cos B\,(n^2\sin^2 C-l^2\sin^2 A)+ n\cos C\,(l^2\sin^2 A-m^2\sin^2 B)=0,$$

or the locus of Q is the cubic

$$x\cos A\,(b^2y^2-c^2z^2)+y\cos B\,(c^2z^2-a^2x^2)+z\cos c\,(a^2x^2-b^2y^2)=0,$$

which may easily be seen to pass through A, B, C, the centre of gravity G, the intersection of perpendiculars, and A′, B′, C′, the angular points of the triangle formed by drawing lines through A, B, C parallel to the opposite sides.

The tangents at A, B, C, meet on the curve in T the intersection of perpendiculars.

The tangents at A′, B′, C′, G meet in a point on the curve.

1693. (Proposed by Dr. BOOTH, F.R.S.)—Let $\Pi(m,\omega)$, $\Pi(m,\phi)$, $\Pi(m,\psi)$ be three arcs of a parabola measured from the vertex; ω, ϕ, and ψ their amplitudes (that is, the inclinations to the axis of the focal perpen-

diculars on the tangents at the extremities of the arcs, being connected by the equation $\tan\omega = \tan\phi\sec\psi + \sec\phi\tan\psi$. Show that their algebraic sum is equal to the product of the ordinates at their extremities divided by the square of the semi-parameter; and apply the theorem to the particular case in which $\tan\omega = 2$, $\tan\phi = \frac{1}{2}$, $\tan\psi = \frac{1}{2}\sqrt{5}$.

1729. (Proposed by Dr. BOOTH, F.R.S.)—Prove that the difference between any parabolic arc (measured from the vertex) and its ordinate is equal to the arc cut off by a focal chord, the amplitudes being connected by the condition $\tan\omega = \sec\phi + \operatorname{cosec}\phi$.

I. *Solution by* J. DALE; E. FITZGERALD; E. MCCORMICK; *and others.*

(1693.) Putting $4m$ for the parameter of the parabola, and r for the radius vector of an arc whose amplitude is θ, we have $r = m\sec^2\theta$,

$$\therefore\ ds = \left\{(dr)^2 + (rd\,.\,2\theta)^2\right\}^{\frac{1}{2}} = 2m\,(1+\tan^2\theta)^{\frac{1}{2}}\,d\,.\tan\theta;$$

$$\therefore\ \Pi\,(m,\omega) = m\left\{\frac{\sin\omega}{\cos^2\omega} + \log\left(\frac{1+\sin\omega}{\cos\omega}\right)\right\},\ \&c.;$$

$$\therefore\ \Pi\,(m,\omega) - \Pi\,(m,\phi) - \Pi\,(m,\psi) = m\left\{\frac{\sin\omega}{\cos^2\omega} - \left(\frac{\sin\phi}{\cos^2\phi} + \frac{\sin\psi}{\cos^2\psi}\right)\right\}$$

$$+\,m\log\left\{\frac{1+\sin\omega}{\cos\omega} \div \left(\frac{1+\sin\phi}{\cos\phi}\right).\left(\frac{1+\sin\psi}{\cos\psi}\right)\right\}.$$

But from the equation of condition we have

$$\frac{1}{\cos\omega} = \frac{1+\sin\phi\sin\psi}{\cos\phi\cos\psi},\quad \frac{1+\sin\omega}{\cos\omega} = \left(\frac{1+\sin\phi}{\cos\phi}\,.\,\frac{1+\sin\psi}{\cos\psi}\right);$$

$$\therefore\ \log\left\{\frac{1+\sin\omega}{\cos\omega} \div \left(\frac{1+\sin\phi}{\cos\phi}\,.\,\frac{1+\sin\psi}{\cos\psi}\right)\right\} = \log\,(1) = 0.$$

We have also $$\frac{\sin\omega}{\cos^2\omega} = \frac{(\sin\phi+\sin\psi)\,(1+\sin\phi\sin\psi)}{\cos^2\phi\cos^2\psi},$$

and $$\frac{\sin\phi}{\cos^2\phi} + \frac{\sin\psi}{\cos^2\psi} = \frac{(\sin\phi+\sin\psi)\,(1-\sin\phi\sin\psi)}{\cos^2\phi\cos^2\psi};$$

$$\therefore\ \frac{\sin\omega}{\cos^2\omega} - \left(\frac{\sin\phi}{\cos^2\phi} + \frac{\sin\psi}{\cos^2\psi}\right) = \frac{2\,(\sin\phi+\sin\psi)\sin\phi\sin\psi}{\cos^2\phi\cos^2\psi}$$

$$= 2\tan\omega\tan\phi\tan\psi.$$

Moreover $y_\omega = 2m\tan\omega$, &c.; $\therefore\ y_\omega y_\phi y_\psi = 8m^3\tan\omega\tan\phi\tan\psi$;

therefore $\Pi\,(m,\omega) - \Pi\,(m,\phi) - \Pi\,(m,\psi) = y_\omega y_\phi y_\psi \div (2m)^2$.

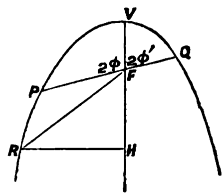
In the particular case given, the equation of condition is satisfied; and the algebraical sum is $m\sqrt{5}$.

(1729.) This may be obtained from the foregoing solution by supposing $\psi = \frac{1}{2}\pi - \phi$, when we have

$$\Pi\,(m,\omega) - \Pi\,(m,\phi) - \Pi\,(m,\psi) = 2m\tan\omega = y_\omega,\ \text{or}$$

$$\Pi\,(m,\omega) - y_\omega = \Pi\,(m,\phi) + \Pi\,(m,\tfrac{1}{2}\pi-\phi) = \text{arc of focal chord of amplitude } \phi.$$

II. *Solution by* W. S. B. WOOLHOUSE



Let VP be any arc of a parabola, vertex V, focus F, and equation $y^2 = 4mx$. Then if 2ϕ denote the polar angle VFP, we shall have

$$x = m \tan^2 \phi,\quad y = 2m \tan \phi\,;$$

and
$$s_\phi = \sqrt{x(m+x)} + m \log \frac{\sqrt{x} + \sqrt{(m+x)}}{\sqrt{m}}$$
$$= m\left(\frac{\sin\phi}{\cos^2\phi} + \log\frac{1+\sin\phi}{\cos\phi}\right)$$
$$= m\left(\frac{\sin\phi}{\cos^2\phi} + \frac{1}{2}\log\frac{1+\sin\phi}{1-\sin\phi}\right) \quad \text{.................................(1).}$$

Therefore, if $2\phi'$ be the polar angle of any second arc, the sum of the two arcs will be

$$s_\phi + s_{\phi'} = m\left(\frac{\sin\phi}{\cos^2\phi} + \frac{\sin\phi'}{\cos^2\phi'}\right) + \frac{m}{2}\log\frac{(1+\sin\phi)(1+\sin\phi')}{(1-\sin\phi)(1-\sin\phi')} \quad \text{............(2).}$$

Let us compare this value with the length of another parabolic arc having 2ω as polar angle, viz.,

$$s_\omega = m\frac{\sin\omega}{\cos^2\omega} + \frac{m}{2}\log\frac{1+\sin\omega}{1-\sin\omega} \quad \text{.........................(3).}$$

Suppose the transcendental or final terms of (2) and (3) to be equal; then

$$\frac{1+\sin\omega}{1-\sin\omega} = \frac{(1+\sin\phi)(1+\sin\phi')}{(1-\sin\phi)(1-\sin\phi')} \quad \text{........................(4),}$$

which gives

$$\sin\omega = \frac{\sin\phi + \sin\phi'}{1+\sin\phi\sin\phi'},\quad \cos\omega = \frac{\cos\phi\cos\phi'}{1+\sin\phi\sin\phi'},\quad \tan\omega = \frac{\sin\phi+\sin\phi'}{\cos\phi\cos\phi'} \quad \text{...}(\alpha)\,;$$

also,
$$\frac{\sin\phi}{\cos^2\phi} + \frac{\sin\phi'}{\cos^2\phi'} = \frac{\sin\phi\,(1-\sin^2\phi') + \sin\phi'\,(1-\sin^2\phi)}{\cos^2\phi\cos^2\phi'}$$
$$= \frac{(\sin\phi + \sin\phi')(1-\sin\phi\sin\phi')}{\cos^2\phi\cos^2\phi'} = \tan\omega\,\frac{1-\sin\phi\sin\phi'}{\cos\phi\cos\phi'}$$

and
$$\frac{\sin\omega}{\cos^2\omega} = \frac{\tan\omega}{\cos\omega} = \tan\omega\,\frac{1+\sin\phi\sin\phi'}{\cos\phi\cos\phi'}\,;$$

$$\therefore\quad \frac{\sin\omega}{\cos^2\omega} - \left(\frac{\sin\phi}{\cos^2\phi} + \frac{\sin\phi'}{\cos^2\phi'}\right) = 2\tan\omega\tan\phi\tan\phi' \quad \text{...............(5).}$$

Hence, by virtue of (4) and (5), we find that

$$\left.\begin{aligned} s_\omega - (s_\phi + s_{\phi'}) &= 2m\tan\omega\tan\phi\tan\phi' \\ \text{or,}\quad s_\phi + s_{\phi'} &= s_\omega - y_\omega \tan\phi\tan\phi' \end{aligned}\right\} \quad \text{................}(\beta).$$

The formulæ (α), (β) enable us to evaluate the sum of any two arcs of a parabola by means of a single arc.

In the same manner may be determined analogous formulæ which will enable us to evaluate the difference of any two arcs of a parabola by means of a single arc. We need, however, only to change the algebraic sign of ϕ',

since the formula (2) will obviously then express $s_\phi - s_{\phi'}$. Thus putting $-\phi'$ for ϕ, the relations (α) and (β) become

$$\left.\sin\omega' = \frac{\sin\phi - \sin\phi'}{1-\sin\phi\sin\phi'},\ \cos\omega' = \frac{\cos\phi\cos\phi'}{1-\sin\phi\sin\phi'},\ \tan\omega' = \frac{\sin\phi - \sin\phi'}{\cos\phi\cos\phi'}\right\}\ \ldots(\gamma)$$

$$\left.\begin{array}{ll} & (s_\phi - s_{\phi'}) - s_{\omega'} = 2m\tan\omega'\tan\phi\tan\phi' \\ \text{or,} & s_\phi - s_{\phi'} = s_{\omega'} + y_{\omega'}\tan\phi\tan\phi' \end{array}\right\}\ \ldots\ldots\ldots\ldots(\delta).$$

To obtain the particular case stated in the Quest. 1729, let the two arcs VP, VQ be determined by a common focal chord PQ, as shown in the diagram; then $\phi+\phi' = \frac{1}{2}\pi$, and $\tan\phi\tan\phi' = 1$;

$$\therefore\ s_\phi + s_{\phi'} = s_\omega - y_\omega,\quad s_\phi - s_{\phi'} = s_{\omega'} + y_{\omega'},$$

and by (α) and (γ) these neat properties respectively hold good when the amplitudes, or semi-polar angles ω, ω' are determined by the formulæ

$$\tan\omega = \sec\phi + \operatorname{cosec}\phi,\quad \tan\omega' = \sec\phi - \operatorname{cosec}\phi.$$

The former of these, in which we have necessarily $\sec\omega > 3$, is the property enunciated in the question. In the latter, ω' may have any value.

Other curious properties may be deduced from the foregoing general relations. As another example, let $\phi = \phi'$; then

$$\tan\omega'' = \frac{2\sin\phi}{\cos^2\phi}\ \text{and}\ 2s_\phi = s_{\omega''} - y_{\omega''}\tan^2\phi.$$

We may add that, according to (4) the formulæ (α) may be replaced by the corresponding relation

$$\tan(45° + \tfrac{1}{2}\omega) = \tan(45° + \tfrac{1}{2}\phi)\tan(45° + \tfrac{1}{2}\phi')\ \ldots\ldots\ldots\ldots(\alpha),$$

and that, from similar considerations, the formulæ (γ) may be replaced by

$$\tan(45° + \tfrac{1}{2}\omega') = \frac{\tan(45° + \frac{1}{2}\phi)}{\tan(45° + \frac{1}{2}\phi')}\ \ldots\ldots\ldots\ldots(\gamma).$$

These would be more convenient for logarithmic computation.

2114. (Proposed by Rev. J. Blissard.)—Prove that $\tan(\cos\theta) =$

$$\frac{\dfrac{\cos\theta}{1} - \dfrac{\cos 3\theta}{1.2.3} + \dfrac{\cos 5\theta}{1.2.3.4.5} - \&c.}{1 - \dfrac{\cos 2\theta}{1.2} + \dfrac{\cos 4\theta}{1.2.3.4} - \&c.} = \frac{\dfrac{\sin 2\theta}{1.2} - \dfrac{\sin 4\theta}{1.2.3.4} + \dfrac{\sin 6\theta}{1.2.3.4.5.6} - \&c.}{\dfrac{\sin\theta}{1} - \dfrac{\sin 3\theta}{1.2.3} + \dfrac{\sin 5\theta}{1.2.3.4.5} - \&c.}$$

Solution by S. W. Bromfield; D. M. Anderson; *and many others.*

Since $2\cos\theta = \epsilon^{i\theta} + \epsilon^{-i\theta}$ and $2i\sin\theta = \epsilon^{i\theta} - \epsilon^{-i\theta}$, where as usual i stands for $\sqrt{(-1)}$, the *first* fraction becomes, after reduction

$$= \frac{\sin\epsilon^{i\theta} + \sin\epsilon^{-i\theta}}{\cos\epsilon^{i\theta} + \cos\epsilon^{-i\theta}} = \tan\tfrac{1}{2}(\epsilon^{i\theta} + \epsilon^{-i\theta}) = \tan(\cos\theta).$$

The *second* fraction may, in like manner, be also reduced to $\tan(\cos\theta)$.

1787. (Proposed by Professor CREMONA.)—On donne une conique K et un point p. Une transversale menée arbitrairement par p rencontre K en deux points m, m'; et soit x un point de la transversale tel que le rapport anharmonique $(pxmm')$ soit un nombre λ donné. Trouver le lieu du point x. Si λ est l'une des racines cubiques imaginaires de -1, on a une certaine conique C (p). De quelle manière change C (p), si l'on fait varier p ?

Recherche analogue par rapport à une surface du second ordre.

Solution by H. R. GREER, B.A.

Take polar coordinates, the point p being the origin, and let the equation of the conic K so referred be $Æ\rho^2 + S\rho + C = 0$; where, for Cartesian coordinates, $Æ = A\cos^2\theta + B\sin^2\theta + 2F\sin\theta\cos\theta$, $S = 2(E\cos\theta + D\sin\theta)$, and C = absolute term. Let the roots of this be ρ_1, ρ_2; form the equation whose root, r, is submitted to the condition $\dfrac{\rho_1(\rho_2 - r)}{\rho_2(r-\rho_1)} = \lambda$, this will be the locus, say L. We can see à priori that it is, generally, of the second degree, and will be unchanged if $\dfrac{1}{\lambda}$ be put for λ. In fact, the equation of L is

$$\left\{(1-\lambda)^2\,CÆ + \lambda S^2\right\}r^2 + (1+\lambda)^2\,CSr + (1+\lambda)^2\,C^2 = 0.$$

If we write $Ær^2 + Sr + C = K$, and $Sr + 2C = P$; so that $K=0$ is the equation of K, and $P=0$ that of the polar of p with regard to K; it may be thrown into the form $(1-\lambda)^2\,CK + \lambda P^2 = 0$, whence it appears that L has double contact with K at the points of contact of tangents drawn from p. Assume $\lambda^2 + 1 = \mu\lambda$, μ being any real constant; then L may be written $(\mu-2)\,CK + P^2 = 0$; that is to say, L may be real though λ be imaginary, and will be so if the *modulus* of $\lambda\,(=\alpha + i\beta)$ be unity. This happens for the imaginary cube-roots of ± 1. The radius vector will meet L in real points if $(1-\lambda)^2 S^2 - 4(\overline{1-\lambda})^2\,CÆ > 0$, that is, meets K and L in real points simultaneously if λ be real, and non-simultaneously if λ be unreal, a result which the equation of the locus *ought* to exhibit.

A precisely similar investigation holds for conicoids. L is determined by λ, and, conversely, λ is determined by L, K and p being given.

1872. (Proposed by Professor CAYLEY.) — Show that the surfaces $xyz = 1$, $yz + zx + xy + x + y + z + 3 = 0$, intersect in two distinct cubic curves; and find the equations of the cubic cones which have their vertices at the origin and pass through these curves respectively.

Solution by SAMUEL ROBERTS, M.A.

By elimination of z, we get the conditions

$$\left(1 + x + \frac{1}{y}\right)\left(1 + y + \frac{1}{x}\right) = 0,\ z = \frac{1}{xy};$$

hence the coordinates of points on the curve of intersection must be of the forms

$$x,\ -\frac{1+x}{x},\ -\frac{1}{1+x}\ \ldots(1),\quad x,\ -\frac{1}{1+x},\ -\frac{1+x}{x}\ldots(2).$$

Substituting these values in the equation of a plane, we have cubics to determine x. Therefore the curves represented are cubics.

Points belonging to the system or curve (1) lie on the cubic cone

$$z^2y+y^2x+x^2z-3xyz=0;$$

those belonging to the system (2) lie on the cubic cone

$$z^2x+x^2y+y^2z-3xyz=0.$$

It is easy to see that if (a, b, c) satisfy the given equations, they are also satisfied by the six permutations of (a, b, c) and the six permutations of $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$. For a given value of x, three of the direct permutations belong to (1), and three belong to (2). In the case of the inverse permutations the order is reversed.

1892. (Proposed by Professor SYLVESTER.)—Observing that the form

$$\left\{(b^2-a^2)x+bcy+c^2z\right\}^2-2d^2\left\{(b^2+a^2)x+bcy+c^2z\right\}z+d^4z^2$$

is a function solely of $x, y, z, a^2-b^2, c^2-d^2, bc, ad$; show that if

$$\alpha^2=\frac{b^2-a^2}{c^2-d^2}d^2,\quad \beta^2=\frac{b^2-a^2}{c^2-d^2}c^2,\quad \gamma^2=\frac{c^2-d^2}{b^2-a^2}b^2,\quad \delta^2=\frac{c^2-d^2}{b^2-a^2}a^2,$$

and A, B, B′ be three points in a straight line such that $AB=\frac{c}{b}$, $AB'=\frac{\gamma}{\beta}$; then, if any point P be found satisfying the equation $a.AP+b.BP=d$, on giving right signs to α, β, the equation $\alpha.AP+\beta.B'P=\delta$ will also be satisfied.

Solution by W. H. LAVERTY; S. W. BROMFIELD; *and others.*

We see that $a^2-b^2=\alpha^2-\beta^2$, $c^2-d^2=\gamma^2-\delta^2$, $bc=\beta\gamma$, $ad=\alpha\delta$;

$\therefore\ F(x, y, z, a^2-b^2, c^2-d^2, bc, ad)=F(x, y, z, \alpha^2-\beta^2, \gamma^2-\delta^2, \beta\gamma, \alpha\delta).$

Let $$AP^2=\frac{x}{z}=\frac{(-y)^2}{4z^2},$$

then $$BP^2=(AP-AB)^2=\left(\frac{-y}{2z}-\frac{c}{b}\right)^2=\frac{x}{z}+\frac{c^2}{b^2}+\frac{cy}{bz}.$$

Now $$a^2.AP^2+b^2.BP^2+2ab.AP.BP-d^2=0;$$

$$\therefore\ (a^2+b^2)\frac{x}{z}+b^2\left(\frac{c^2}{b^2}+\frac{cy}{bz}\right)+2ab\left(\frac{-y}{2z}-\frac{c}{b}\right)\left(\frac{-y}{2z}\right)-d^2=0;$$

therefore $$(b^2+a^2)x+c^2z+bcy-d^2z=-a(2bx+cy);$$

therefore $$\left\{(b^2+a^2)x+c^2z+bcy-d^2z\right\}^2=4a^2(b^2x^2+bcxy+c^2xz),$$

or $\quad \left\{(b^2-a^2)\,x+bcy+c^2z\right\}^2-2d^2\left\{(b^2+a^2)\,x+bcy+c^2z\right\}z+d^4z^2=0;$

$\therefore$ also $\left\{(\beta^2-a^2)\,x+\beta\gamma y+\gamma^2z\right\}^2-2\delta^2\left\{(\beta^2+a^2)x+\beta\gamma y+\gamma^2z\right\}z+\delta^4z^2=0;$

therefore $\quad (\beta^2+a^2)\,x+\beta\gamma y+\gamma^2z-\delta^2z=-a\,(2\beta x+\gamma y);$

therefore $a^2\,.\,\dfrac{x}{z}+\beta^2\left\{\dfrac{x}{z}+\dfrac{\gamma}{\beta}\,.\,\dfrac{y}{z}+\dfrac{\gamma^2}{\beta^2}\right\}+2a\beta\left(\dfrac{x}{z}\right)^{\frac{1}{2}}\left\{\left(\dfrac{x}{z}\right)^{\frac{1}{2}}+\dfrac{\gamma}{\beta}\right\}=\delta^2;$

therefore $\quad a^2\,.\,\mathrm{AP}^2+\beta^2\,(-\mathrm{AP}+\mathrm{B'A})^2+2a\beta\,.\,\mathrm{AP}\,.\,(\mathrm{AP}-\mathrm{B'A})=\delta^2;$

therefore $\quad a\,.\,\mathrm{AP}+\beta\,.\,\mathrm{BP'}=\delta.$

1969. (Proposed by Professor SYLVESTER.)—In two given great circles of a sphere intersecting at O are taken respectively two points P and Q, the arc joining which is of given length: prove that S, H two fixed points, and M a fixed line, in a plane may be found such that, for all positions of the arc PQ, a point M in the fixed line may be found satisfying the equations

$$\mathrm{SM} \pm \mathrm{HM} = \sin \mathrm{OP}, \quad \mathrm{SM} \mp \mathrm{HM} = \sin \mathrm{OQ}.$$

Solution by PROFESSOR CAYLEY.

1. In the spherical triangle OPQ, whereof the sides OP, OQ, PQ are θ, ϕ, β and the angle O is $= a$, the relation between these quantities is $\cos a = \dfrac{\cos\beta - \cos\theta\cos\phi}{\sin\theta\sin\phi}$; hence treating a, β as constants, and θ, ϕ as variable angles connected by the foregoing equation, it is required to show that we can find two fixed points S, H and a fixed line, such that taking M a variable point in this line and writing $\mathrm{SM} = r$, $\mathrm{HM} = s$, the relation between r and s (or equation of the fixed line in terms of r, s as coordinates of a point thereof) is obtained by substituting in the foregoing equation for θ and ϕ the values given by the two equations

$$\sin\theta = (r+s), \quad \sin\phi = (r-s),$$

or, as for the sake of homogeneity, it will be more convenient to write these equations, $\qquad m\sin\theta = (r+s), \quad m\sin\phi = (r-s).$

2. Suppose that the perpendicular distances of S, H from the fixed line are a and b, and that the distance between the feet of the two perpendiculars is $2c$, then if x denote the distance of the point M from the midway point between the feet of the two perpendiculars, we have

$$r = \sqrt{\left\{(c+x)^2+a^2\right\}}, \quad s = \sqrt{\left\{(c-x)^2+b^2\right\}},$$

and (a, b, c) being properly determined, the elimination of x from these equations should give between (r, s) a relation equivalent to that obtained by the elimination of (θ, ϕ) from the before mentioned equations. Or, what is the same thing, the elimination of (r, s, x) from the equations

$m \sin\theta = r+s,\quad m\sin\phi = r-s,\ r=\sqrt{\{(c+x)^2+a^2\}},\ s=\sqrt{\{(c-x)^2+b^2\}}$
should give between (θ, ϕ) the relation

$\cos\alpha = \dfrac{\cos\beta-\cos\theta\cos\phi}{\sin\theta\sin\phi}$; that is, the last mentioned equation should be obtained by the elimination of x from the equations

$m(\sin\theta+\sin\phi)=2\sqrt{\{(c+x)^2+a^2\}},\ m(\sin\theta-\sin\phi)=2\sqrt{\{(c-x)^2+b^2\}}.$

3. The equation in (θ, ϕ) may be written

$$\cos\beta-\cos\alpha\sin\theta\sin\phi = \cos\theta\cos\phi,$$

or squaring and reducing

$$\sin^2\theta+\sin^2\phi = \sin^2\beta+2\cos\alpha\cos\beta\sin\theta\sin\phi+\sin^2\alpha\sin^2\theta\sin^2\phi,$$

that is,

$$\sin^2\theta+\sin^2\phi = \frac{1-\cos^2\alpha-\cos^2\beta}{\sin^2\alpha}+\left(\sin\alpha\sin\theta\sin\phi+\frac{\cos\alpha\cos\beta}{\sin\alpha}\right)^2.$$

But from the two equations in x, we have

$$m^2(\sin^2\theta+\sin^2\phi) = 4c^2+2a^2+2b^2+4x^2,\quad m^2\sin\theta\sin\phi = 4cx+a^2-b^2,$$

whence
$$2x = \frac{b^2-a^2+m^2\sin\theta\sin\phi}{2c},$$

therefore $\sin^2\theta+\sin^2\phi = \dfrac{4c^2+2b^2+2a^2}{m^2}+\left(\dfrac{b^2-a^2+m^2\sin\theta\sin\phi}{2cm}\right)^2.$

Hence, comparing the two results, we have

$$\frac{1-\cos^2\alpha-\cos^2\beta}{\sin^2\alpha} = \frac{4c^2+2b^2+2a^2}{m^2},\quad \frac{\cos\alpha\cos\beta}{\sin\alpha} = \frac{b^2-a^2}{2cm},\quad \sin\alpha = \frac{m}{2c};$$

or, as these may also be written,

$$\sin\alpha = \frac{m}{2c},\ \cos^2\alpha+\cos^2\beta = \frac{-b^2-a^2}{2c^2},\ 2\cos\alpha\cos\beta = \frac{b^2-a^2}{2c^2};$$

whence $\quad (\cos\alpha+\cos\beta)^2 = \dfrac{-a^2}{c^2},\ (\cos\alpha-\cos\beta)^2 = \dfrac{-b^2}{c^2},\ \sin\alpha=\dfrac{m}{2c};$

so that m being put equal to unity, or otherwise assumed at pleasure, a, b, c are given functions of α, β. Or conversely, if a, b, c are assumed at pleasure, then α, β, m are given functions of these quantities.

5. It is to be remarked that (α, β) being real, a and b will be imaginary, and consequently the points S, H of Professor SYLVESTER's theorem are imaginary;* if, however, we write $-a^2$, $-b^2$ in place of a^2, b^2 respectively, then the radicals $\sqrt{\{(c+x)^2-a^2\}}$, $\sqrt{\{(c-x)^2-b^2\}}$ have a real geometrical interpretation. The system of relations between $(\alpha, \beta, a, b, c, m)$ becomes

$$(\cos\alpha+\cos\beta)^2 = \frac{a^2}{c^2},\ (\cos\alpha-\cos\beta)^2 = \frac{b^2}{c^2},\ \sin\alpha = \frac{m}{2c};$$

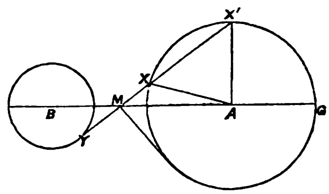
and considering (a, b, c) as given, we may write

$$\cos\alpha = \frac{a+b}{2c},\ \cos\beta = \frac{a-b}{2c},\ m = \sqrt{\{4c^2-(a+b)^2\}},$$

[* Prof. SYLVESTER remarks that according as β is less or greater than α, we may find real values of θ, ϕ equal to one another in the one case and supplementary in the other. Hence we must in any case be able to make $r = 0$ and $s = 0$ indifferently, which shows *à priori* that the line being supposed real, each point S, H must be imaginary, but so that the squared distance of either from the line must be a *real negative quantity*, conformably to Prof. CAYLEY's important observation in the text.]

viz., we have either this system or the similar system obtained by writing $-b$ in place of b.

6. Consider two circles with the radii a, b and having the distance of their centres $= 2c$, and to fix the ideas assume that $2c > a + b$, that is, that the circles are exterior to each other. The foregoing equations signify that $90^\circ - \alpha$, $90^\circ - \beta$ are the inclinations to the line of centres of the inverse and the direct common tangents respectively, and that m is the length of the inverse common tangent. And the theorem is, that considering two circles as above, and taking M a variable point in the line of centres, if r, s denote the tangential distances of m from the two circles respectively, and if m be the length of the inverse common tangent of the two circles, then the angles θ, ϕ determined by the equations



$$m \sin\theta = r + s, \quad m \sin\phi = r - s,$$

are connected by the relation

$$\cos\beta = \cos\theta \cos\phi + \sin\theta \sin\phi \cos\alpha,$$

(α, β) being constant angles, determined as above.

7. It is to be remarked that, assuming $k = \dfrac{\sin\alpha}{\sin\beta} = \dfrac{\sqrt{\{4c^2 - (a+b)^2\}}}{\sqrt{\{4c^2 - (a-b)^2\}}}$,

that is, $k =$ inverse common tangent $\div$ direct common tangent, then we have $\cos\alpha = \sqrt{(1 - k^2 \sin^2\beta)} = \Delta\beta$, or the equation in θ, ϕ becomes

$$\cos\beta = \cos\theta \cos\phi + \sin\theta \sin\phi \Delta\beta,$$

which is the algebraical equation connecting the amplitudes of the elliptic functions in the relation $\mathrm{F}(\theta) + \mathrm{F}(\phi) = \mathrm{F}(\beta)$.

8. It is very noticeable that the above figure leads to another relation in elliptic functions, viz., it is the very figure employed for that purpose by JACOBI; in fact, considering therein YM as a variable tangent meeting the circle A in the two points X and X′, then if 2ψ, $2\psi'$ denote the angles GAX, GAX′ respectively, it is easy to see geometrically that we have $d\psi : d\psi' = \mathrm{YX} : \mathrm{YX}'$; where $(\mathrm{YX})^2 = (\mathrm{BX})^2 - b^2 = 4c^2 + a^2 + 4ac\cos 2\psi - b^2 = (2c+a)^2 - b^2 - 8ac\sin^2\psi$, and similarly $(\mathrm{YX}')^2 = (2c+a)^2 - b^2 - 8ac\sin^2\psi'$, that is, writing $l^2 = \dfrac{8ac}{(2c+a)^2 - b^2}$, the differential equation is

$$\frac{d\psi}{\sqrt{(1 - l^2\sin^2\psi)}} - \frac{d\psi'}{\sqrt{(1 - l^2\sin^2\psi')}} = 0,$$

corresponding to an integral equation $\mathrm{F}(\psi) - \mathrm{F}(\psi') = \mathrm{F}(\mu)$,

the modulus of the elliptic functions being $\quad l, = \dfrac{\sqrt{8ac}}{\sqrt{\{(2c+a)^2 - b^2\}}}$.

In the problem above considered the modulus is $k, = \dfrac{\sqrt{\{4c^2 - (a+b)^2\}}}{\sqrt{\{4c^2 - (a-b)^2\}}}$,

and it is not very easy to see the connexion between the two results.

1940. (Proposed by the Rev. J. BLISSARD.)—
Given that $1.2.3\ldots x$ (x inf.) $\sqrt{(2\pi)}\; x^{x+\frac{1}{2}}\,\epsilon^{-x}$, prove that

$$m(m+n)\ldots\Big\{m+(x-1)\,n\Big\}\ (x\text{ inf.}) = \frac{\sqrt{(2\pi)}}{\Gamma\left(\frac{m}{n}\right)}\, n^x x^{x+\frac{m}{n}-\frac{1}{2}}\,\epsilon^{-x}.$$

Solution by the PROPOSER.

In $1.2.3\ldots.x\ (x\text{ inf.}) = \sqrt{(2\pi)}\,x^{x+\frac{1}{2}}\epsilon^{-x}$ put $x+m-1$ for x, and divide by $1.2\ldots.(m-1)$, which $=\Gamma(m)$; then we have

$$m(m+1)\ldots.(m+x-1) = \frac{\sqrt{(2\pi)}}{\Gamma(m)}\,(x+m-1)^{x+m-\frac{1}{2}}\,\epsilon^{-(x+m-1)}.$$

Taking logarithms of each side, we have

$$\log\Big\{m(m+1)\ldots.(m+x-1)\Big\}$$
$$= \log\left\{\frac{\sqrt{(2\pi)}}{\Gamma(m)}\right\} + (x+m-\tfrac{1}{2})\left(\log x + \frac{m-1}{x} + \&\text{c.}\right) - (x+m-1)$$
$$= \log\left\{\frac{\sqrt{(2\pi)}}{\Gamma(m)}\right\} + (x+m-\tfrac{1}{2})\log x - x;$$

therefore $$m(m+1)\ldots.(m+x-1) = \frac{\sqrt{(2\pi)}}{\Gamma(m)}\,x^{x+m-\frac{1}{2}}\epsilon^{-x};$$

and putting $\frac{m}{n}$ for m, and multiplying by n^x, we have

$$m(m+n)\ldots.\Big\{m+(x-1)\,n\Big\} = \frac{\sqrt{(2\pi)}}{\Gamma\left(\frac{m}{n}\right)}\,n^x\,x^{x+\frac{m}{n}-\frac{1}{2}}\epsilon^{-x}.$$

1990. (Proposed by Professor SYLVESTER.)—Prove that the three points in which a circular cubic is cut by any transversal are the foci of a Cartesian oval passing through the four foci of the cubic.

Solution by M. W. CROFTON, B.A.

This singular theorem, of which Professor CAYLEY has given an instructive discussion by rectangular coordinates on pp. 35—39 of Vol. VI. of the *Reprint*, admits of a simple proof depending entirely upon the focal properties of the curves.

I. Given three points on a Cartesian oval 1, 2, 3, and one focus F, to find the locus of the two other foci.

Let ρ_1, ρ_2, ρ_3 be the distances of 1, 2, 3 from F, and $\sigma_1, \sigma_2, \sigma_3$ their distances from G, another focus: we shall have

$$\rho_1 + k\sigma_1 = l,\quad \rho_2 + k\sigma_2 = l,\quad \rho_3 + k\sigma_3 = l;$$

eliminate k, l, and we find

$$(\rho_2-\rho_3)\,\sigma_1+(\rho_3-\rho_1)\,\sigma_2+(\rho_1-\rho_2)\,\sigma_3=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

showing clearly that the locus required is a circular cubic passing through F, and having 1, 2, 3 as concyclic foci.

Also, *if* 1, 2, 3 *are any three points on a Cartesian, its foci lie on some circular cubic, of which* 1, 2, 3 *are concyclic foci.*

II. Suppose now that 1, 2, 3, 4 are the four concyclic foci of a circular cubic; let A, B be any two points on the curve, draw a Cartesian having A, B as two foci, passing through 1 and 2 (there is but one such Cartesian); let $\rho_1, \rho_2, \rho_3, \rho_4$ be the vectors from A to 1, 2, 3, 4; $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ from B. Suppose the equation of the Cartesian to be

$$a\rho+b\sigma=1\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Since (ρ_1, σ_1) and (ρ_2, σ_2) both satisfy this equation, the point (ρ_3, σ_3) will also satisfy it; for by the (focal) definition of the circular cubic

$$\rho_3=\frac{m\rho_1+n\rho_2}{m+n},\quad \sigma_3=\frac{m\sigma_1+n\sigma_2}{m+n};$$

and these values evidently satisfy (2).

Hence the Cartesian passes through the focus 3 of the cubic, and also through 4.

Therefore, taking any two points A, B on the circular cubic, a Cartesian can be drawn with A, B as two foci, and passing through the 4 concyclic foci of the cubic. The third focus of the Cartesian will be the point in which AB produced meets the cubic again, as is evident from (I). This proves Professor SYLVESTER's theorem.

The following remarkable property follows at once from the above:—*If three points* A, B, C *on a Cartesian be given, and one focus F, the family of Cartesians all pass through a 4th fixed point.* This point will be the 4th focus of the circular cubic through F with A, B, C as foci. For, from (I.), the three foci of the Cartesian lie on this cubic, and therefore from (II.) it is clear that it passes through the 4th focus of the cubic, as well as through A, B, C.

1994. (Proposed by M. W. CROFTON, B.A.)—Two circles have double internal contact with an ellipse, and a third circle passes through the four points of contact. If t, t', T be the tangents from any point on the ellipse to these three circles, prove that $T^2 = t\,t'$.

I. *Solution by the* REV. R. TOWNSEND, F.R.S.

If the equations of the three circles be respectively $s=0$, $s'=0$, and $S=0$, then that of the ellipse, as touching s and s' at their intersections with S, being necessarily $ss'=S^2$, therefore, &c.

N.B.—That the equation $ss'=S^2$ should reduce to the second order, when s, s', and S are any three circles, the centre of S must be collinear with and lie midway between those of s and s'; hence, if a conic have double contact with two circles, the centre of the circle determined by the four points of contact lies midway between those of the touched circles.

II. *Solution by* W. H. LAVERTY; H. TOMLINSON; J. DALE; W. CHADWICK; *and others*.

Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, be the equation to the ellipse; $(d, 0)$ and $(x'\ y')$ the coordinates respectively of the centre, and one point of contact of one of the circles; r its radius: then its equation is $(x-d)^2+y^2 = r^2$, and therefore the equation of the common tangent must be of both the forms

$$(x-d)(x'-d)+yy' = r^2, \text{ and } \frac{xx'}{a^2} + \frac{yy'}{b^2} = 1;$$

$$\therefore\ \frac{a^2}{x'}(x'-d) = b^2 = r^2 + d(x'-d);\ \therefore\ x' = \frac{d}{e^2},\ \text{and } r^2 = b^2 - \frac{b^2d^2}{a^2-b^2}.$$

The length of the tangent to this circle from the point $\left(x, \frac{b}{a}\sqrt{(a^2-x^2)}\right)$ is given by

$$t^2 = (x-d)^2 + \frac{b^2}{a^2}(a^2-x^2) - b^2 + \frac{b^2d^2}{a^2-b^2} = \left(ex - \frac{d}{e}\right)^2.$$

Similarly we find that $\quad t' = ex - \frac{d'}{e}$.

$$\text{Now } T^2 = \left(x - \frac{d+d'}{2}\right)^2 + \frac{b^2}{a^2}(a^2-x^2) - \left\{\left(\frac{d+d'}{2} - \frac{d}{e^2}\right)^2 + b^2 - \frac{b^2}{a^2}\left(\frac{d}{e^2}\right)^2\right\}$$

$$= e^2x^2 - (d+d')x + \frac{dd'}{e^2} = \left(ex - \frac{d}{e}\right)\left(ex - \frac{d'}{e}\right)$$

$= t\,.\,t'$; which proves the theorem.

ON SOME PROBLEMS IN THE THEORY OF CHANCES.
BY HUGH GODFRAY, M.A.

THE following remarks may serve to show why discordant results are sometimes obtained in treating questions of probability, such as the four-point problem (*Reprint*, Vol. V., p. 81).

I believe it will be found that the discordance arises from the fact, that the word *random* is not sufficiently defined in the question; and the possibility of considering it in different ways, makes so many different problems of which the various results are solutions.

Perhaps my meaning will be made clearer by an example, and the following will answer the purpose:—Two chords are drawn at random in a circle, what is the chance that they will intersect?

Now, what is a chord drawn at random?

(1.) We may consider the circumference of the circle to be divided into a great number of very small equal arcs, and that by a chord drawn at random we mean a chord joining any two of these arcs,—*all combinations being equally probable.*

(2.) We may consider that a random chord means a line whose distance

from the centre is less than the radius,—*all such distances being equally probable.*

(3.) A random chord may mean any line joining two points of the circle, *all lengths less than the diameter being equally probable.*

&c. &c.

It will be easy to show that the average length of such chords in Case (2) will be greater than in (1), but that in (3) they will be much less.

To continue the discussion of the problem:—Two such chords are said to be drawn at random. Here we have another element of uncertainty introduced, which can only be removed by some further limitation. The limitation which is usually, but tacitly, supposed, is that the second chord may be inclined at any angle to the first, and that *all inclinations are equally probable*; but it would not be difficult to give other meanings to the random relative positions of the two chords.

Now assuming all inclinations of the two chords to be equally probable, it will be found that the chance of the two chords intersecting will be $\frac{1}{3}$, when the interpretation of a random chord is that of Case (1); it will be $\frac{1}{2}$ in Case (2); and $\dfrac{3\pi-8}{4\pi}$ in Case (3).

Case (1) is the view Mr. Woolhouse has taken in his solution of the more general Question 1894. (*Reprint*, Vol. V., p. 110.)

Case (2) was set by Professor Adams in the *Smith's Prize Examination*, in 1865.

Case (3) I have not met with before, and as a simple exercise in integration it is proposed for solution as Question 2263.

I shall now consider the solutions of the four-point problem on p. 81, Vol. V., and it will be seen that Mr. Wilson and Dr. Ingleby have solved two different problems.

Taking a point at random, according to Dr. Ingleby, is supposing the unlimited area to be divided into an infinite number of indefinitely small *equal* areas, and assuming that the point is equally likely to occupy any one of these; so that the chance of its falling within any given area δ is $\dfrac{\delta}{\text{whole area}}$. This is a satisfactory definition, and if legitimately followed out will give $\frac{1}{2}$ for the probability, because $\epsilon+\phi-\delta$ is indefinitely small compared with $\alpha+\beta+\gamma$.

If ABC (Fig. 1) be the triangle formed by joining three of the points; and, δ being the area of the triangle, if we call α the infinite area contained between the lines AB and AC indefinitely produced &c., the probability will be

$$\frac{\alpha+\beta+\gamma+\delta}{2(\alpha+\beta+\gamma)-2\delta}=\frac{1}{2}+\frac{\delta}{\alpha+\beta+\gamma-\delta}=\frac{1}{2},$$

since δ is indefinitely small compared with $\alpha+\beta+\gamma$.

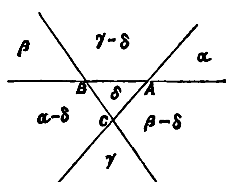


FIG. I.

Mr. Wilson's definition of a random point is, that it is the intersection of two random lines; and from this definition he has correctly obtained the value $\frac{1}{3}$.

The special law according to which these random lines are drawn will not affect the result.

If we suppose two points A and B (Fig. 2) to have been determined, and that a third point C is required by the intersection of two random lines through A and B. We may suppose round each of the points an indefinite number of these random lines, and the small quadrilateral spaces formed may be considered as each equally likely to be occupied by a random point. But these areas are clearly not equal, and therefore a random point according to this definition is not the same as according to the former, and the probability may therefore be expected to be different.

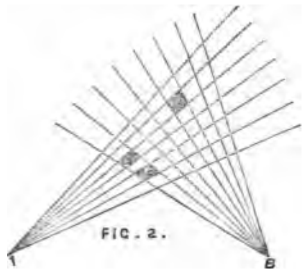


FIG. 2.

I have no doubt that the various other solutions referred to by Dr. INGLEBY may be reconciled in the same say.

1996. (Proposed by W. K. CLIFFORD.)—If four circles $A = 0$, $B = 0$, $C = 0$, $D = 0$ are mutually orthotomic, the square of the radius of a circle $lA + mB + nC + sD = 0$ is $(l^2r_1^2 + m^2r_2^2 + n^2r_3^2 + s^2r_4^2) \div (l+m+n+s)^2$, where r_1, r_2, r_3, r_4 are the radii of A, B, C, D.

I. *Solution by* W. H. LAVERTY; H. TOMLINSON; W. CHADWICK; *and others.*

If (α_1, β_1), (α_2, β_2) be the centres of two of the circles, we must have

$$(\alpha_2-\alpha_1)^2 + (\beta_2-\beta_1)^2 = r_2^2 + r_1^2 \quad\ldots\ldots\ldots\ldots (\alpha),$$

and five other similar conditions.

Now dividing $lA + mB + nC + sD = 0$ by $(l+m+n+s)$, and writing it in the form $(x-\alpha)^2 + (y-\beta)^2 = r^2$, we find

$$r^2 = \frac{\{\Sigma(l\alpha_1)\}^2 + \{\Sigma(l\beta_1)\}^2}{\{\Sigma(l)\}^2} + \frac{\Sigma(lr_1^2) - \Sigma\big(l(\alpha_1^2+\beta_1^2)\big)}{\Sigma(l)}$$

$$= \frac{\Sigma(l^2r_1^2) + \Sigma\big(lm\left[2(\alpha_1\alpha_2+\beta_1\beta_2) + r_2^2 + r_1^2 - (\alpha_1^2+\beta_1^2+\alpha_2^2+\beta_2^2)\right]\big)}{\{\Sigma(l)\}^2}.$$

But by the condition (α) the second term of the numerator vanishes,

therefore $$r^2 = \frac{l^2r_1^2 + m^2r_2^2 + n^2r_3^2 + s^2r_4^2}{(l+m+n+s)^2}.$$

II. *Solution by* J. DALE; H. MURPHY; *and others.*

Let the equations of the circles be

(A) .... $(x-a_1)^2 + (y-b_1)^2 - r_1^2 = 0$
(B) .... $(x-a_2)^2 + (y-b_2)^2 - r_2^2 = 0$
(C) .... $(x-a_3)^2 + (y-b_3)^2 - r_3^2 = 0$
(D) .... $(x-a_4)^2 + (y-b_4)^2 - r_4^2 = 0$

If these circles are mutually orthotomic, we must have the following conditions :—

$$(a_1-a_2)^2+(b_1-b_2)^2 = r_1^2+r_2^2 \dots (1) \quad (a_2-a_3)^2+(b_2-b_3)^2 = r_2^2+r_3^2 \dots (4)$$
$$(a_1-a_3)^2+(b_1-b_3)^2 = r_1^2+r_3^2 \dots (2) \quad (a_2-a_4)^2+(b_2-b_4)^2 = r_2^2+r_4^2 \dots (5)$$
$$(a_1-a_4)^2+(b_1-b_4)^2 = r_1^2+r_4^2 \dots (3) \quad (a_3-a_4)^2+(b_3-b_4)^2 = r_4^2+r_4^2 \dots (6)$$

The equation of the circle $lA+mB+nC+sD = 0$ is

$$(l+m+n+s)(x^2+y^2)-2(la_1+ma_2+na_3+sa_4)x-2(lb_1+mb_2+nb_3+sb_4)y + [l(a_1^2+b_1-r_1^2)+m \text{ \&c.}] = 0,$$

and the square of the radius of this circle is equal to

$$\frac{(la_1+ma_2+na_3+sa_4)^2+(lb_1+mb_2+nb_3+sb_4)^2}{(l+m+n+s)^2} - \frac{(l+m+n+s)\left\{l(a_1^2+b_1^2-r_1^2)\right\}+\text{\&c.}}{(l+m+n+s)^2}$$

where the coefficients of l^2, m^2, n^2, s^2 are respectively r_1^2, r_2^2, r_3^2, r_4^2; also the coefficient of any of the other terms, such as lm, is

$$r_1^2+r_2^2+2(a_1a_2+b_1b_2)-(a_1^2+b_1^2+a_2^2+b_2^2)$$
$$= r_1^2+r_2^2-\left\{l(a_1-a_2)^2+(b_1-b_2)^2\right\} = 0, \text{ by (1)}.$$

Similarly the coefficients of ln, ls, &c., vanish; hence the expression for the square of the radius reduces to the form given in the question.

2001. (Proposed by W. GODWARD.)—If r and r_1 be the radii of two circles each having double contact with a conic, the former passing through the centre of the conic, and the latter through one of the foci; prove that

$$r : r_1 = a : 2b.$$

Solution by J. DALE; W. CHADWICK; H. TOMLINSON; W. H. LAVERTY *and others.*

Taking the case of the ellipse; the circles of double contact, passing, the one through the centre, and the other through a focus, must touch the ellipse internally, and have their centres on the major axis. Taking the origin at the centre, let (x, y) be the point of contact of the circle (r) with the conic; then we have

$$r^2 = (\text{normal})^2 = y^2+(1-e^2)^2x^2 = (1-e^2)(a^2-e^2x^2) = e^4x^2,$$

therefore $\quad a^2-e^2x^2 = a^2e^2$, and $r^2 = a^2e^2(1-e^2) = e^2b^2$.

Again, let (x, y) be the point of contact of the circle (r_1) with the conic; then

$$r_1^2 = (\text{normal})^2 = (1-e^2)(a^2-e^2x^2) = e^2(a-ex)^2,$$

therefore $(a^2-e^2x^2) = 4a^2e^2(1-e^2)$ and $r_1^2 = 4a^2e^2(1-e^2)^2 = 4e^2\dfrac{b^4}{a^2}$,

therefore $\quad r : r_1 = a : 2b.$

In the case of the hyperbola, the value for r^2 shows that the construction is impossible. The circle in this case must touch the hyperbola externally and have its centre on the minor axis, so that the above equation does not remain true. If, however, we take the circle r as the one having double contact with the *conjugate* hyperbola, and passing through the centre, we shall still have the relation $r : r_1 = a : 2b$.

II. *Solution by the* PROPOSER.

It is evident from the symmetry of the curves that the centres of the two circles will be in the direction of the transverse axis of the conic; taking therefore the centre of the conic for the origin, and its principal diameters as axes of coordinates, the equation of the circle whose radius is r is

$$y^2-2rx+x^2=0 \ldots\ldots (1);$$

and the coordinates of the centre of the circle whose radius is r_1 are $(ae \mp r_1, 0)$, so that its equation is

$$x^2+y^2-2(ae \mp r_1)x+a^2e^2 \mp 2aer_1=0 \ldots. (2),$$

the upper signs being taken when the conic is an ellipse, and the lower when an hyperbola.

Also the equation to the ellipse and *conjugate* hyperbola is

$$\frac{b^2x^2}{a^2} \pm y^2 \mp b^2=0 \ldots\ldots (3),$$

and the equation to the ellipse and hyperbola

$$\frac{b^2x^2}{a^2} \pm y^2 - b^2=0 \ldots\ldots (4).$$

Eliminating y^2 between (1) and (3), we have

$$e^2x^2-2rx+b^2=0;$$

hence in order that (1) and (3) should have contact we must have $r^2=b^2e^2$, or $r=be$.

Likewise eliminating y^2 between (2) and (4), we have

$$e^2x^2-2(ae \mp r_1)x+a^2e^2 \mp 2aer_1 \pm b^2=0,$$

$$\text{or,} \quad e^2x^2-2(ae \mp r_1)x+a^2 \mp 2aer_1=0 \ldots\ldots (5),$$

so that for contact of (2) and (4), we must have

$$(ae \mp r_1)^2=e^2(a^2 \mp 2aer_1), \text{ which gives } r_1=\pm 2ae(1-e^2)=\frac{2b^2e}{a};$$

therefore $$r : r_1 = a : 2b.$$

COR.—Let P (x, y) be the point of contact of (2) and (4); then if we substitute $r_1=\pm 2ae(1-e^2)$ in (5), we shall finally obtain $x=\frac{a}{e}(2e^2-1)$; also from (4),

$$y^2=\pm b^2\left(1-\frac{x^2}{a^2}\right)=\frac{a^2(1-e^2)}{e^2}(-1+5e^2-4e^4)=\frac{a^2}{e^2}(-1+6e^2-9e^4+4e^6).$$

Let F be the focus through which circle radius r_1 passes, then its coordinates are $(ae, 0)$. These coordinates of F and P, give FP $=\pm 2a(1-e^2)=$ latus rectum of the conic. This corollary furnishes a Solution to a question in the Senate House Examinations for 1865, where it was proposed for the ellipse. The property also holds for the parabola.

1865. (Proposed by P. O'CAVANAGH.)—Find the equation and parameter of the parabola osculating most closely at the origin the conic

$$ax + by + cx^2 + 2dxy + ey^2 = 0 \ldots\ldots (1);$$

and find also the angle (θ) between the axis of x and the axis of the required parabola.

Solution by the REV. J. L. KITCHIN, M.A.; S. W. BROMFIELD; W. H. LAVERTY; *and others.*

1. The equation of the required parabola may be written

$$ax + by + h^2x^2 + 2hkxy + k^2y^2 = 0 \ldots\ldots (2),$$

$ax + by = 0$ being the common tangent at the origin to (1) and (2), and $hx + ky = 0$ the diameter of (2) through the origin.

Subtracting (1) from (2) we get

$$(h^2 - c)\, x^2 + 2\,(hk - d)\, xy + (k^2 - e)\, y^2 = 0 \ldots\ldots (3),$$

which must represent a pair of common chords of (1) and (2) passing through the origin. Now in order that the parabola (2) may have the closest possible (4-pointic) contact with the given conic (1), the equation (3) must be identical with $(ax + by)^2 = 0$, hence we must have

$$h^2 - c = la^2,\quad hk - d = lab,\quad k^2 - e = lb^2;\quad \therefore\ (la^2 + c)(lb^2 + e) = (lab + d)^2,$$

whence $\quad l = \dfrac{ce - d^2}{\mathrm{D}}$, where $\mathrm{D} = 2abd - a^2e - b^2c$; and then we have

$$h^2 = \frac{(ad - bc)^2}{\mathrm{D}},\ \text{and}\ k^2 = \frac{(ae - bd)^2}{\mathrm{D}}.$$

Hence the equation to the osculating parabola at the origin is

$$\left\{(ad - bc)\, x + (ae - bd)\, y\right\}^2 = \mathrm{D}\,(ax + by).$$

2. *Otherwise:* the equation to the tangent at the origin being $ax + by = 0$, the equation to the parabola must be

$$k\,(ax + by + cx^2 + 2dxy + ey^2) - (ax + by)^2 = 0;$$

and that this may be a parabola, we must have

$$(kd - ab)^2 = (kc - a^2)(ke - b^2),\ \text{whence}\ k = \frac{2dab - (a^2e + b^2c)}{d^2 - ce};$$

and the equation to the parabola is as given in Art. 1.

3. Now $(ad - bc)\, x + (ae - bd)\, y = 0$ is the equation to the diameter of the parabola through the origin, and the diameter is parallel to the axis; therefore we have $\quad \tan\theta = \dfrac{ad - bc}{bd - ae}$.

4. We easily find the equation to the directrix; it is

$$(ae - bd)\, x - (ad - bc)\, y = \tfrac{1}{4}\,(a^2 + b^2).$$

The perpendicular from the origin on this line is equal to the focal distance of the origin, and the parameter (p') of the diameter through the origin is four times this distance,

therefore $\quad p' = \dfrac{a^2 + b^2}{\sqrt{\left\{(ad - bc)^2 + (ae - bd)^2\right\}}}$.

The tangent at the origin to the parabola is $ax + by = 0$.

If ϕ be the angle between this line and the diameter of the parabola, then the latus rectum or principal parameter $p = p' \sin^2 \phi$.

We find, $\sin^2 \phi = \dfrac{D^2}{(a^2+b^2)\left\{(ad-bc)^2+(ae-bd)^2\right\}}$,

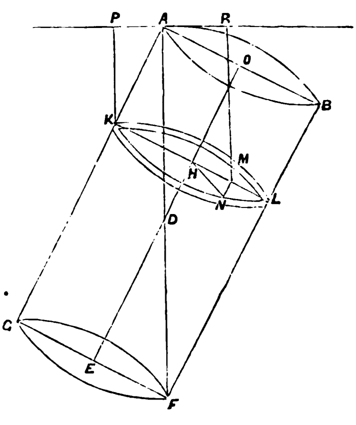
therefore $$p = \frac{D^2}{\left\{(ad-bc)^2+(ae-bd)^2\right\}^{\frac{3}{2}}}.$$

It is now easy to find the coordinates of the focus and vertex of the parabola; and consequently the equation to the axis of the parabola.

[From the above value for $\tan\theta$ it is clear that the axis of the parabola is parallel to the diameter of the given conic through the point of contact. The same property is proved in the *Lady's and Gentleman's Diary* for 1859 (pp. 65—70), where it was proposed to find "the locus of the focus of a varying parabola osculating most closely a given ellipse or hyperbola, at the various points of the given conic."]

1674. (Proposed by J. W. T. Blakemore, B.A.)—A cylinder filled with fluid is closed at both ends, and then suspended from a point in the rim of one end; find the resultant pressure on the curved surface, and prove that the direction of this resultant pressure, and the axis of the cylinder, make equal angles with the vertical and horizontal respectively.

Solution by the Rev. J. L. Kitchin, M.A.



Let A be the point of suspension of the cylinder. It is evident that the centre of gravity of the cylinder of fluid will be at D the middle point of its axis, and that the line AD will be vertical. PAR is the trace of the vertical plane AGFB on the horizontal plane through A.

Let $AO = r$, $OE = 2h$, $\alpha = \angle ADO = \tan^{-1}\dfrac{r}{h}$ (a known angle), $\angle LHN = \theta$, and $AK = x$.

Small element at $N = rd\theta dx$; $\therefore$ pressure on it $= g\rho\, rd\theta dx \,.\, RN = g\rho rd\theta dx\left\{x\cos\alpha + (r + r\cos\theta)\sin\alpha\right\}$. Since it acts normally to the

surface, its direction is through H the centre of the section. The resolved part along LK is $= g\rho r d\theta dx \left\{ x \cos \alpha + (r + r \cos \theta) \sin \alpha \right\} \cos \theta$;
therefore the resolved pressure along LK from M and N is

$$= 2g\rho r d\theta dx \left\{ x \cos \alpha + (r + r \cos \theta) \sin \alpha \right\} \cos \theta.$$

The resolved pressures perpendicular to KL are equal and opposite, and therefore destroy each other.

Therefore the pressure (perpendicular to FB) on the whole curved surface

$$= 2g\rho \int_0^{2h} \int_0^{\pi} dx \left\{ x \cos \alpha \cos \theta \, d\theta + r \sin \alpha \cos \theta \, d\theta + r \sin \alpha \cos^2 \theta \, d\theta \right\}$$

$$= 2g\rho r \int_0^{2h} dx \,.\, \frac{\pi}{2} r \sin \alpha = 2g\rho\pi r^2 h \sin \alpha$$

= the resultant pressure on the curved surface.

The axis of the cylinder makes an angle with the horizontal $= \frac{1}{2}\pi - \alpha =$ the angle LK makes with AD, which proves the proposition in the question.

[We add here a solution of an analogous question which was proposed for investigation in our review of Besant's *Hydrostatics*, in the *Educational Times* for May, 1864.

A hollow cone without weight, closed and filled with water, is suspended from a point in the rim of its base; prove that (1), if α be the semivertical angle of the cone, the total pressures on the curved surface and the base are in the ratio $(1 + 11 \sin^2 \alpha)$: $12 \sin^3 \alpha$; and (2), if ϕ be the angle which the direction of the resultant pressure on the curved surface makes with the vertical, $\cot \phi = \frac{1}{48} (\cot^3 \alpha + 28 \cot \alpha)$.

1. Let B, C be the respective areas of the base and curved surface of the cone; b, c the depths of their centres of gravity below the point of suspension; P_b, P_c the whole pressures of the fluid on these areas; and w the weight of a unit of volume; then $P_b = Bbw$, and $P_c = Ccw$. Now the centres of gravity of the volume and surface of the cone are at the respective distances of three-fourths and two-thirds of the axis from the vertex; also the point of suspension and the centre of gravity of the volume of the cone must be in the same vertical line; hence, putting a for the radius of the base of the cone, and β for the inclination of the axis to the vertical, we have $B = \pi a^2$, and $C = \pi a^2 \operatorname{cosec} \alpha$; also

$$a \cot \beta = \tfrac{1}{4} a \cot \alpha, \text{ or } \tan \beta = 4 \tan \alpha;\ b = a \sin \beta;\ c = a (\sin \beta + \tfrac{1}{3} \cot \alpha \cos \beta);$$

$$\therefore\ P_b : P_c = b : c \operatorname{cosec} \alpha = \sin \alpha : 1 + \tfrac{1}{3} \cot \alpha \cot \beta = 12 \sin^3 \alpha : 1 + 11 \sin^2 \alpha.$$

2. The resultant vertical pressure on the whole surface is equal to the weight of the fluid, that is to $\frac{1}{3}\pi a^3 w \cot \alpha$; also the vertical component of the whole pressure on the base is $P_b \cos \beta = \pi a^3 w \sin \beta \cos \beta$; therefore the resultant vertical pressure (V) on the curved surface is

$$V = \pi a^3 w \left(\tfrac{1}{3} \cot \alpha + \sin \beta \cos \beta\right).$$

The horizontal pressure on the curved surface is equal to the horizontal component (H) of the whole pressure on the base; therefore

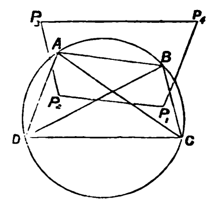
$$H = \pi a^3 w \sin^2 \beta.$$

Hence $\cot \phi = \dfrac{V}{H} = \frac{1}{3} \cot \alpha \, (1 + \cot^2 \beta) + \cot \beta$

$$= \tfrac{1}{3} \cot \alpha \left(1 + \tfrac{1}{16} \cot^2 \alpha\right) + \tfrac{1}{4} \cot \alpha = \tfrac{1}{48} (\cot^3 \alpha + 28 \cot \alpha).]$$

1986. (Proposed by J. GRIFFITHS, M.A.)—Given four points on a circle: it is required to show that the "polar centres" of the four triangles that can be formed from them lie on another circle of equal radius.

Solution by J. DALE; S. W. BROMFIELD; *and others.*



Taking the "polar centre" of a triangle to signify the centre of the circle with respect to which the triangle is self-conjugate, then the "polar centre" is identical with the intersection of the perpendiculars.

Let A, B, C, D be the four points, and P_1, P_2, P_3, P_4 the polar centres of the triangles BCD, CDA, DAB, ABC respectively. Then, as AP_2 and BP_1 are perpendicular to DC and equal to each other, therefore P_1P_2 is equal and parallel to AB, so also P_2P_3, P_3P_4, P_4P_1 are respectively equal and parallel to BC, CD, DA.

Hence the quadrilateral $P_1P_2P_3P_4$ is equal in all respects to the quadrilateral ABCD, and consequently the circle through $P_1P_2P_3P_4$ is equal to the circle through ABCD.

NOTE.—It may be readily shown that the lines AP_1, BP_2, CP_3, DP_4 meet in one point, and hence that the quadrilateral $P_1P_2P_3P_4$ is obtained by turning ABCD through an angle of 180° round this point.

1844. (Proposed by W. H. LAVERTY.)—If $(\epsilon)_1$ represent ϵ^x, $(\epsilon)_2$ represent ϵ^{ϵ^x}, &c.; and if $\log^2(x)$ represent $\log_\epsilon(\log_\epsilon x)$, &c.; find the value of

$$\int_0^\infty \frac{(\epsilon)_{n-1}\cdot(\epsilon)_{n-2}\cdot\dots(\epsilon)_1\cdot dx}{(\epsilon)_n\cdot\sqrt{(\epsilon)_{n-1}}};\quad \int_0^\infty \frac{\log^n(x)\,.\,dx}{\left\{\log^{n-1}(x)\right\}^2.\log^{n-2}(x)..\log(x).x}$$

Solution by the PROPOSER.

Let $(\epsilon)_{n-1}=y$; then $(\epsilon)_n=\epsilon^y$, and $(\epsilon)_{n-1}\cdot(\epsilon)_{n-2}\cdot\dots(\epsilon)_1\cdot dx = dy$; therefore the first integral is equal to

$$\int^\infty \frac{dy}{(\epsilon)_n\cdot\sqrt{(\epsilon)_{n-1}}} = \int^\infty \epsilon^{-y}y^{-\frac{1}{2}}dy = \Gamma(\tfrac{1}{2}) = \sqrt{(\pi)}.$$

2. Let $\log^n(x) = y$; then $\log^{n-1}(x) = \epsilon^y$, and

$$\frac{dx}{\log^{n-1}x)\,.\,\log^{n-2}(x)\dots\log(x)\,.\,x} = dy;$$

therefore the second integral is equal to

$$\int_0^\infty \epsilon^{-y}y\,dy = \Gamma(2) = 1.$$

THEOREM : BY PROFESSOR CAYLEY.

If (A, A′), (B, B′) are four points (two real and the other two imaginary) related to each other as foci and antifoci, (that is, if the lines AA′, BB′ intersect at right angles in a point O in such wise that $OA = OA' = i.OB = i.OB'$), then the product of the distances of any point P from the points A, A′ is equal to the product of the distances of the same point P from the two points B, B′.

In fact, the coordinates of A, A′ may be taken to be $(a, 0)$, $(-a, 0)$, and those of B, B′ to be $(0, ai)$, $(0, -ai)$; whence, if (x, y) are the coordinates of P, we have

$$(AP)^2 = (x-a)^2+y^2 = (x-a+iy)(x-a-iy)$$
$$(A'P)^2 = (x+a)^2+y^2 = (x+a+iy)(x+a-iy)$$
$$(BP)^2 = x^2+(y-ia)^2 = (x+iy+a)(x-iy-a)$$
$$(B'P)^2 = x^2+(y+ia)^2 = (x+iy-a)(x-iy+a),$$

from which the theorem is at once seen to be true.

An important application of the theorem consists in the means which it affords of passing from the foci (A, B, C, D) of a bicircular quartic, to the antifoci (A, B) and (C, D); viz., if these are (A′, B′, C′, D′), then the equation $l\sqrt{(A)}+m\sqrt{(B)}+n\sqrt{(C)} = 0$, must be transformable into $l'\sqrt{(A')}+m'\sqrt{(B')}+n'\sqrt{(C')} = 0$. Writing these respectively under the forms

$$l^2A+m^2B-n^2C+2lm\sqrt{(AB)} = 0, \quad l'^2A'+m'^2B'-n'^2C'+2l'm'\sqrt{(A'B')}=0,$$

the two radicals $\sqrt{(AB)}$, $\sqrt{(A'B')}$ are identical; and the remaining terms in the two equations respectively are rational functions, which when the ratios $l' : m' : n'$ are properly determined will be to each other in the ratio $lm : l'm'$; the two equations being thus identical.

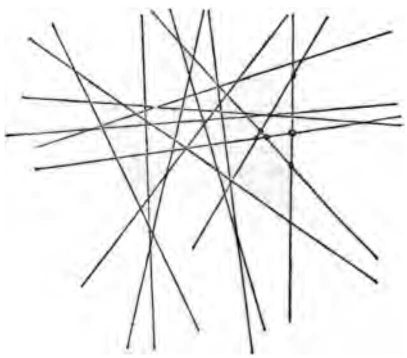
NOTE ON QUESTION 1894. BY W. S. B. WOOLHOUSE, F.R.A.S.

On p. 72 of this volume of the *Reprint* there is a paper by Mr. GODFRAY, "On some Problems in the Theory of Chances." His remarks do not anticipate the statements I have yet to make on the various solutions to Professor SYLVESTER's problem of four points. Mr. GODFRAY, by way of illustration, has referred to my solution to the general problem, Quest. 1894 (*Reprint*, Vol. V., p. 110), and has ventured to give three separate and distinct meanings of "a chord drawn at random." As this may be calculated to mislead, I must really set the matter right. Only one legitimate meaning can attach to the phrase in question, and at the outset of the solution referred to I have proved that a random chord may indiscriminately and with equal probability enter into all the combinations stated in Mr. GODFRAY's Case (1). This, therefore, is not an arbitrary hypothesis, but a matter of fact inherent in the investigation. The Cases (2) and (3) adduced by Mr. GODFRAY are however quite artificial assumptions, which may do very well for the manufacture of mathematical exercises, but they cannot be accepted as correct definitions, or such as are in any way consistent with our elementary notions of the simple meaning of a random chord. There can be no doubt whatever that the only true meaning is that which is adopted in my solution.

On the Four-point Problem. By J. M. Wilson, M.A., F.G.S.

Mr. Godfray's remarks (*Reprint,* Vol. VI., p. 72), tempt me to contribute a few words towards the discussion of the solutions of this problem.

The solution of the problem requires (1) that the definition of a random point be exact and correct, (2) that it be correctly argued from. Now there can be no doubt that the correct definition of a random point is that it is one equally likely to fall in any one of the small equal areas into which we suppose space to be divided. Dr. Ingleby's definition (*Reprint,* Vol. V., p. 82,) is perfectly correct. But Mr. Woolhouse's remarks (*Reprint,* Vol. VI., p. 49) show very clearly that the solution is wrong; because it implies that the triangle formed by the first three points is finite, while the fourth point, and the fourth point only, has an infinite range. In fact δ is not infinitely small compared with $\alpha+\beta+\gamma$, the assumption on which his solution depends. My own solution is of a different kind. I will here reproduce it in an altered form.



Draw a number of intersecting lines as in the figure, at random, that is, according to no law (for to speak of random lines drawn according to a special law, with Mr. Godfray, is to me unintelligible). They determine by their intersection a determinate number of points. These points may be grouped in fours; (1) completely, taking every four; (2) partially, taking fours which are all formed by four lines; excluding in both cases the groups in which more than two lie on the same line.

The first method is not a basis for calculation, so far as I can see.

Adopting the second, I observe that, grouping the lines by fours, I get for every four lines three groups of four points, in which one group always forms a reentrant quadrilateral, two groups always convex ones. This enables me to say, that of all the groups of four points which lie on four lines in the figure, one-third form reentrant quadrilaterals. Or, in other words, if a finite number of lines be drawn on a sheet of paper large enough to admit of their intersection, and four points be taken at random among the points of their intersection, such that the four always lie on four lines, the probability of the four forming a reentrant quadrilateral is $\frac{1}{3}$. Up to this point I imagine there can be no dispute. To proceed. Let the number of the lines be increased very greatly. The points of intersection will spread uniformly over space, that is to say, in equal areas there will ultimately be equal numbers of them. And this will be equally true whether we take the complete or partial grouping of the points. Hence the result $\frac{1}{3}$ is true for any finite number of lines, and therefore for an infinite number of lines; and *when the number of lines is infinite the points will satisfy the definition of random points,* that is, they will be equally distributed over space.

I may add, that when the four points are taken within any triangle, the result has been independently and unquestionably shown to be $\frac{1}{3}$.

NOTE ON THE GEOMETRY OF THE TRIANGLE. BY J. GRIFFITHS, M.A.

If P denote the point of intersection of the three perpendiculars, and G the centroid of any triangle ABC; then, as we know, the following circles are coaxal; viz., 1, The nine-point circle, the circumscribing and self-conjugate circles; 2, The circle described upon PG as diameter (see *Reprint*, Vol. II., p. 25). The object of the present Note is to point out another circle of the system in question.

Let S denote the circle circumscribing the triangle; S′ the self-conjugate circle; and S″ the circle circumscribing the triangle A′B′C′, formed by the tangents to S at the vertices A, B, C; then the circles S, S′, S″ are coaxal.

For, taking the triangle ABC as triangle of reference, and adopting the ordinary notation, we easily find the equations to the vertices of the triangle A′B′C′ to be $\frac{\alpha}{-a} = \frac{\beta}{b} = \frac{\gamma}{c}$, $\frac{\alpha}{a} = \frac{\beta}{-b} = \frac{\gamma}{c}$, $\frac{\alpha}{a} = \frac{\beta}{b} = \frac{\gamma}{-c}$;

and that to the circle S″, which passes through these three points, to be

$$a\cos A \,.\, \alpha^2 + b\cos B \,.\, \beta^2 + c\cos C \,.\, \gamma^2 + \lambda\,(a\beta\gamma + b\gamma\alpha + c\alpha\beta) = 0,$$

where $\lambda = \dfrac{a^3\cos A + b^3\cos B + c^3\cos C}{abc} = 1 + 4\cos A\cos B\cos C.$

Hence the theorem in question. Hence also, if α, β, γ be the points of contact of the sides of the triangle ABC with one of the four circles (I) which can be drawn to touch them; then the nine-point circle of the triangle $\alpha\beta\gamma$, and the circles S and I, are coaxal.

1878. (Proposed by W. K. CLIFFORD.)—A line of length a is broken up into n pieces at random; prove that (1) the chance that they cannot be made into a polygon of n sides is $n2^{1-n}$; and (2) the chance, that the sum of the squares described on them does not exceed

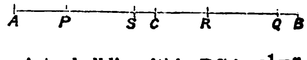
$$\frac{a^2}{n-1}, \text{ is } \left(\frac{\pi}{n^2-n}\right)^{\frac{1}{2}(n-1)} \frac{\Gamma(n)}{\Gamma\left\{\frac{1}{2}(n+1)\right\}} \cdot \frac{1}{n^{\frac{1}{2}}}.$$

Solution by the PROPOSER.

1. Let us define as follows. A point is taken *at random* on a (finite or infinite) straight line, when the chance that the point lies on a finite portion of the line varies as the length of that portion. And, a line is broken up at random when the points of division are taken at random.

Now, the n pieces will always be capable of forming a polygon except when one of them is greater than the sum of all the rest; that is, greater than half the line. The *first* part of the question may therefore be stated thus; $n-1$ points are taken at random on a finite line; to find the chance that some one of the intervals shall be greater than half the line.

2. *First* Solution. Bisect the line AB at C. Then the chance that one of the points of division shall lie within BC is



$\frac{1}{2}$; therefore the chance that all the $n-1$ points shall lie within BC is 2^{1-n}.

But this is the chance that the *first piece* (reckoning from A) shall be greater than AC. Next, I say that the chance of the rth piece being greater than half the line is equal to the chance of the $(r+1)$th piece being greater. For, let PQ be the portion which is made up of the rth and $(r+1)$th pieces. And take PR = QS = AC. Then if the point of division between the rth and $(r+1)$th pieces lies within RQ, the rth piece is greater than AC; and if it lies within PS, the $(r+1)$th piece is greater than AC. But RQ = PS; therefore by definition the chances are equal. Consequently, the chance that any one of the n pieces shall be greater than AC is equal to the chance that any other of the n pieces shall be greater than AC. And all these n events are mutually exclusive; while we have proved that the chance of the first of them is 2^{1-n}. Therefore the chance that some one piece is greater than AC is $n2^{1-n}$.

3. *Second* Solution. I am convinced that there is a fallacy in the above, and have therefore tried to get a rigorous proof in this way. Take P a point in AC, and let AP $=x$. Consider a small element dx at P. I want to find the chance that the rth piece, reckoning from A, may begin at P (within the element dx) and be greater than AC. This requires, first, that one of the $n-1$ points of division shall be within dx; the chance of this is $(n-1)\dfrac{dx}{a}$; next, $r-2$ of the remaining points must be within AP, and the chance of this is $\dfrac{\underline{|n-2}}{\underline{|n-r}\,\underline{|r-a}}\left(\dfrac{x}{a}\right)^{r-2}$; lastly, the $n-r+1$ points left must be within RB; whose chance is $\left(\dfrac{1}{2}-\dfrac{x}{a}\right)^{n-r+1}$. Therefore the chance required is $\dfrac{\underline{|n-1}}{\underline{|n-r}\,\underline{|r-2}}\left(\dfrac{x}{a}\right)^{r-2}\left(\dfrac{1}{2}-\dfrac{x}{a}\right)^{n-r+1}\dfrac{dx}{a}$.

Now, if we integrate this with respect to x from 0 to $\frac{1}{2}a$, we shall get the entire chance that the rth piece may be greater than AC. The integral is easily found to be 2^{1-n}. And as there are thus n equal chances, whose events are all mutually exclusive, the chance that some one of these events will happen is $n2^{1-n}$.

4. *Third* Solution. To make this clear, I will state first the previously known analogous solutions in the cases where $n=3$ and $n=4$. When the line is divided into three pieces, call them x, y, z, and take their lengths for the coordinates of a point P in geometry of three dimensions. Then, since

$$x+y+z=a \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and x, y, z are all positive, the point P must be somewhere on the surface of the equilateral triangle determined on the plane (1) by the coordinate planes. Now, consider those points on the triangle for which $x>\frac{1}{2}a$. These are cut off by the plane $x=\frac{1}{2}a$; and it is easy to see that this plane cuts off from one corner of the triangle a similar triangle of ***half the linear dimensions***; and therefore of $\frac{1}{4}$ the area. Now, there are three corners cut off; their joint area is therefore $\frac{3}{4}$ of the area of the triangle; and the chance required is accordingly $\frac{3}{4}$.

When the line is divided into *four* pieces, take the ***first three*** pieces as the coordinates of a point in space. Then we have $x+y+z<a$, and x, y, z all positive; so the point must lie within the content of the tetrahedron bounded by the plane $x+y+z=a$ and the coordinate planes. Now, if $x+y+z<\frac{1}{2}a$, the *fourth piece* must be *greater* than $\frac{1}{2}a$. The points for

which this is the case are cut off by the plane $x+y+z=\frac{1}{2}a$; and it is easily seen as before that this plane cuts off from one corner of the tetrahedron a similar tetrahedron of half the linear dimensions, and therefore of $\frac{1}{8}$ the volume. So also the plane $x=\frac{1}{2}a$ cuts off from another corner a similar tetrahedron of half the linear dimensions. Since therefore there are four corners cut off, their joint volume is $\frac{4}{8}$ or $\frac{1}{2}$ of the volume of the tetrahedron; and the chance required is accordingly $\frac{1}{2}$.

5. Now, consider the analogous cases in geometry of n dimensions. Corresponding to a closed area, and a closed volume, we have something which I shall call a *confine*. Corresponding to a triangle, and to a tetrahedron, there is a confine with $n+1$ corners or vertices, which I shall call a *prime* confine, as being the simplest form of confine. A prime confine has also $n+1$ faces, each of which is, not a plane, but a prime confine of $n-1$ dimensions. Any two vertices may be joined by a straight line, which is an *edge* of the confine. Through each vertex pass n edges. A prime confine may be *regular*, which it is when any three vertices form an equilateral triangle; or *rectangular*, which it is when the edges through some one vertex are all equal and at right angles to one another.

To solve the question for general values of n, we may adopt as a type either of the geometrical solutions given for the cases $n=3$ and $n=4$. First, take the lengths of the n pieces for the coördinates of a point in geometry of n dimensions. Then, since their sum is a, and they are all positive, the point must lie within a certain regular prime confine of $n-1$ dimensions. The supposition that a certain piece is greater than $\frac{1}{2}a$ cuts off from one corner of the confine a similar confine of half the linear dimensions, and therefore of 2^{1-n} times the content. And as there are n corners, their joint content is $n\,2^{1-n}$ times the content of the confine; the chance required is consequently $n\,2^{1-n}$. Or, take the lengths of the *first* $n-1$ *pieces* as the coordinates of a point in geometry of $n-1$ dimensions; the point will then lie within a certain *rectangular* confine of $n-1$ dimensions; and the investigation proceeds as before, the n corners being cut off in the same manner.

6. It will be seen that this *third* solution involves in a geometrical form the assumption of which some sort of proof was given in the *first* solution. Let us make this extension of our fundamental definition:—A point is taken *at random* in a (finite or infinite) space of n dimensions, when the chance that the point lies in a finite portion of the space varies as the content of that portion. The assumption is that when the lengths of the pieces into which a line is broken up are taken as coordinates of a point, then if the line is broken up at random the point is taken at random, and *vice versâ*. The proof of this assumption may be shown to involve a geometrical proposition equivalent to the integration by parts of the differential in Art. (3).

Making this assumption, we may solve the second part of the question by the method of the *third* solution of the first part. I will first state the previously known analogous solution of the case where $n=3$. The question is in this case,—If a line of length a be broken into three pieces at random, find the chance that the sum of the squares of these pieces shall be less than $\frac{1}{2}a^2$. Take the lengths of the three pieces for coordinates x, y, z of a point P in geometry of three dimensions; then, as before, the point must lie somewhere in the area of the equilateral triangle determined on the plane $x+y+z=a$ by the coordinate planes. But if also the sum of the squares of the pieces is less than a certain quantity m^2, then the point P must lie within a certain circle determined on the plane $x+y+z=a$ by the sphere $x^2+y^2+z^2=m^2$. Now, in the case where $m^2=\frac{1}{2}a^2$ this circle is the circle

inscribed in the equilateral triangle; so that the question reduces itself to this one:—

To find, in terms of the area of an equilateral triangle, the area of its inscribed circle.

Now let us go a little further, and consider the case in which $n = 4$. Here we shall have to take a point P in geometry of four dimensions; the point must lie somewhere in the regular tetrahedron determined on the hyper-plane $x+y+z+w=a$ by the coordinate hyper-planes. If also the sum of the squares of the pieces is less than a certain quantity m^2, then the point P must lie within a certain sphere determined on the hyper-plane $x+y+z+w=a$ by the quasi-sphere $x^2+y^2+z^2+w^2=m^2$. In the particular case where m is the perpendicular from the vertex on the base of a rectangular tetrahedron each of whose equal edges is of length a, or $m^2=\frac{1}{3}a^2$, this sphere is the sphere inscribed in the regular tetrahedron. The question is therefore reduced to this one:—

To find, in terms of the volume of a regular tetrahedron, the volume of its inscribed sphere.

Now, a similar reduction holds in the general case; viz., the question can always be reduced to this one:—

To find, in terms of the content of a regular prime confine of $n-1$ dimensions, the content of its inscribed quasi-sphere.

This question I proceed to solve.

7. Let $n-1=p$. *The perpendicular from any vertex on the opposite face of a regular prime confine in p dimensions* $= \left(\frac{p+1}{2p}\right)^{\frac{1}{2}}$. (edge).

For, let O be the vertex in question, OA, OB,.... the p edges through O. Draw through each vertex A a space of $p-1$ dimensions parallel to the face opposite to A. The p spaces thus drawn will intersect in a point P, such that OP is the diagonal of a confine analogous to a parallelogram and to a parallelepiped. Then OP is p times the perpendicular from O on the opposite face of the regular confine; for the perpendicular is the projection of *one* edge at a certain angle, while OP is the projection at the same angle of a broken line consisting of p edges.

We have also

$$\begin{aligned} OP^2 &= OA^2+OB^2+OC^2+\dots +2OA.OB\cos AOB+\dots \\ &= \Sigma.OA^2+\Sigma.OA.OB, \ [\text{since } \cos AOB = \tfrac{1}{2}, \text{ \&c.}], \\ &= \left\{p+\tfrac{1}{2}p(p-1)\right\}.OA^2 = \tfrac{1}{2}p(p+1).OA^2, \end{aligned}$$

therefore $(\text{perpendicular})^2 = \frac{OP^2}{p^2} = \frac{p+1}{2p}.(\text{edge})^2.$

[If the confine were *rectangular*, or all the angles at O right angles, we should have $\cos AOB = 0$, &c.: and so

$$(\text{perpendicular})^2 = \frac{1}{p}(\text{edge})^2 = \frac{a^2}{n-1};$$

which proves that the question *does* always reduce itself to the one now under consideration.]

The content *of a regular prime confine in p dimensions whose edge is a,* is

$$= \frac{a^p}{\underline{|p}}\left(\frac{p+1}{2^p}\right)^{\frac{1}{2}}$$

Suppose this formula true for $p-1$ dimensions; that is, let

$$V_{p-1} = \frac{a^{p-1}}{\underline{|p-1}} \left(\frac{p}{2^{p-1}}\right)^{\frac{1}{2}}.$$

Now, content of confine $= \frac{1}{p} \times$ perpendicular $\times$ content of face, or

$$V_p = \frac{a}{p} \cdot \left(\frac{p+1}{2^p}\right)^{\frac{1}{2}} \cdot V_{p-1} = \frac{a^p}{\underline{|p}} \cdot \left(\frac{p+1}{2^p}\right)^{\frac{1}{2}}.$$

Hence the formula, if true for one value of p, is true for the next; now it can be immediately verified in the case of $p = 1$; therefore it is generally true.

The radius of the inscribed quasi-sphere $= \dfrac{a}{\{2p(p+1)\}^{\frac{1}{2}}}$.

We can divide the regular confine into $p+1$ equal confines, each having the centre of the inscribed quasi-sphere for vertex; and the content of one of these $= \frac{\rho}{p} \times$ content of face; but the sum of them all is equal to the content of the whole confine. Hence $(p+1)\rho =$ perpendicular of confine

$$= a\left(\frac{p+1}{2p}\right)^{\frac{1}{2}}, \text{ or, } \rho = \frac{a}{\{2p(p+1)\}^{\frac{1}{2}}}.$$

The content of the quasi-sphere $= \rho^p \dfrac{\{\Gamma(\frac{1}{2})\}}{\Gamma(\frac{1}{2}p+1)}$.

For it is the value of $\iiint \ldots dx\, dy\, dz \ldots$ the integral being so taken as to give to the variables all values consistent with the condition that $x^2+y^2+z^2+\ldots$ is not greater than ρ^2. (See TODHUNTER'S *Integral Calculus*, Art. 271.) Let C_p denote this content; then

$$C_p = \rho^p \frac{\{\Gamma(\frac{1}{2})\}^p}{\Gamma(\frac{1}{2}p+1)} = \frac{a^p}{(2p^2+2p)^{\frac{1}{2}p}} \cdot \frac{\{\Gamma(\frac{1}{2})\}^p}{\Gamma(\frac{1}{2}p+1)}$$

therefore $$\frac{C_p}{V_p} = \left(\frac{\pi}{p^2+p}\right)^{\frac{1}{2}p} \cdot \frac{\Gamma(p-1)}{\Gamma(\frac{1}{2}p+1)} \cdot \frac{1}{(p+1)^{\frac{1}{2}}}.$$

Restore $n-1$ for p, and we get the answer to the question, namely,

$$\left(\frac{\pi}{n^2-n}\right)^{\frac{1}{2}(n-1)} \cdot \frac{\Gamma(n)}{\Gamma\{\frac{1}{2}(n+1)\}} \cdot \frac{1}{n^{\frac{1}{2}}}.$$

8. The following are applications of the same method.

If a line be broken up at random into n pieces, the chance of an assigned two of them (the pth and qth from one end) being together greater than half the line, is $n\,2^{1-n}$.

If n pieces be cut off at random, one from each of n equal lines, the chance that the pieces cannot be made into a polygon is $\dfrac{1}{\underline{|n-1}}$.

1990. (Proposed by Professor SYLVESTER.)—

(1.) Prove that the locus of one set of foci of all the conics that touch a given circle at two given points, is another circle passing through those points and the centre of the given circle.

(2.) Prove that the three points in which a circular cubic is cut by any transversal are the foci of a Cartesian oval passing through the four foci of the cubic.

(3.) Prove that a circular cubic is the locus of one set of foci of all the conics that can be drawn through four points lying in a circle.

(4.) Prove that, if a circle and straight line be cut by any transversal in three points, these will be the foci of one of a system of Cartesian ovals having double contact with one another at two fixed points. [This last proposition is Mr. CROFTON'S, and may be proved as a particular case of (2)].

Solution by PROFESSOR HIRST.

The following remarks have reference solely to the first and third parts of the Question, and to the mode of transition from the latter to the former.

Let 1, 2, 3, 4 be any four points on a circle whose centre is O, and let α, β, γ be the intersections of the three pairs of opposite sides of the quadrangle which they form. Or more precisely, let

$\overline{41}$ and $\overline{23}$ (or A and A′) intersect in α,

$\overline{42}$ and $\overline{31}$ (or B and B′) intersect in β,

$\overline{43}$ and $\overline{12}$ (or C and C′) intersect in γ.

Now, by the theorem of Desargues, the line-pairs A, A′; B, B′; C, C′ cut the line at infinity in three pairs of points in involution; and to this involution belong the intersections o and o' of the circle (O) (the circular points at infinity), as well as the infinitely distant points s and s' of every conic (S) circumscribed to the quadrangle 1 2 3 4. The double points δ and $\delta_{,}$ of this involution divide every segment thereof harmonically; hence, S being the centre of (S), the angle between its asymptotes Ss, Ss' is divided harmonically by $S\delta$ and $S\delta_{,}$, and the latter are accordingly conjugate diameters of (S). Moreover, δ and δ' being harmonic conjugates relative to o and o', $S\delta$ and $S\delta_{,}$ are at right angles to each other; consequently they are the axes of (S), and we conclude that *the axes of every conic* (S) *circumscribed to a quadrangle* 1 2 3 4 *which is itself inscribed in a circle* (O) *are parallel to two fixed orthogonal lines* $O\delta$, $O\delta_{,}$.

The line-pairs A, A′; B, B′; C, C′ being included amongst the conics of the system to which (S) belongs, we may at once infer that the bisectors of the adjacent angles formed by any one of these pairs are also parallel to the fixed lines $O\delta$, $O\delta_{,}$. (See Professor SYLVESTER'S Question 1950, *Reprint*, Vol. V., p. 105.)

In the system of circumscribed conics, there are two parabolas (P) and ($P_{,}$) (they are the conics which touch the line at infinity in the double points δ and $\delta_{,}$), and since these are the only conics of the system which have infinitely distant centres, we conclude that the *locus of the centres of all conics is an equilateral hyperbola* (H) *whose asymptotes are parallel to* $O\delta$ *and* $O\delta_{,}$. (H) passes obviously through the centres α, β, γ, and O of the line-pairs A, A′; B, B′; C, C′; and of the circle (O).

Let us now consider any line directed towards the infinitely distant double point δ. Exclusive of the parabola (P) there is but one conic (S) of the system which has its centre thereon, since this line is cut in only one point S, at a finite distance, by the hyperbola (H). Consequently, exclusive of δ, which

must be regarded as a focus of the parabola (P), there are but two *conjugate* foci $f, f_{,}$ of the conics of our system which are situated on any line $S\delta$ parallel to one system of axes. Hence *the locus of such foci is a cubic* (Σ) *which has one asymptote parallel to* $O\delta$. This asymptote, in fact, is also an asymptote of the hyperbola (H), for when $S\delta$ coincides with the latter, S and one of the foci f recede to infinity. Moreover *this cubic* (Σ) *is circular*, for when $S\delta$ moves parallel to itself to infinity, the conic (S) which has its centre thereon is the parabola ($P_{,}$) whose conjugate imaginary foci are the circular points o and o'. Lastly, the *circular cubic* (Σ) *passes obviously through the points* α, β, γ, O, and since in each of these points two conjugate foci $f, f_{,}$, on diameters directed towards δ, coincide, *the tangents to the cubic* (Σ) *at the points* α, β, γ, O *are parallel to the fixed line* $O\delta$.

In like manner *the locus of the conjugate foci situated on axes parallel to* $O\delta_{,}$ *is a circular cubic* ($\Sigma_{,}$), *whose real asymptote coincides with the asymptote of the hyperbola of centres* (H) *which is directed towards* $\delta_{,}$, *and whose tangents at the points* α, β, γ, O *are parallel to this asymptote.*

We are now in a position to pass to the first part of our Question, and to examine what becomes of the cubics (Σ) and ($\Sigma_{,}$) when the four points 1, 2, 3, 4 coincide two and two, say in a and a'. In making this transition we may clearly suppose α, β, and γ, which are the angles of a self-conjugate triangle relative to (O), to remain fixed. Let then 1 and 4, as well as 2 and 3, be made to approach so as ultimately to coincide respectively with a and a' on the polar $\beta\gamma$ of α. The line-pair A, A′ will become a pair of tangents αa, $\alpha a'$ and B, B′ and C, C′ (which are still supposed to intersect in β and γ respectively) will ultimately coincide upon the chord of contact aa'. The points δ and $\delta_{,}$ are now the infinitely distant points of αO and aa' and the lines $\alpha O\delta$ and $aa'\delta_{,}$ must be constituent parts of the cubics (Σ) and ($\Sigma_{,}$), respectively; for on each of these lines, say $\alpha O\delta$, five points of the cubic are situated, viz., two coincident in α, two others in O, and a fifth in δ. The remaining constituent of each circular cubic must of course be a circle. In the case of the cubic (Σ) this circle must pass through β and γ and have tangents at these points parallel to αO. Consequently (Σ) *must break up into the line* αO *and the circle whose diameter is* $\overline{\beta\gamma}$. Similarly ($\Sigma_{,}$) *must break up into the line* $\beta\gamma$ *and the circle whose diameter is* $\overline{\alpha O}$, which circle manifestly passes through a and a'.

It should be observed, however, that β and γ are indeterminate so long as the *manner* in which the points 1 and 4, and 2 and 3, approach coincidence is not stated. Consequently (Σ) may be said to resolve itself into the line αO and an *indeterminate circle* having its centre on the line aa' and cutting (O) orthogonally; the reason of this being that any two points whatever equidistant from a line may be regarded as the foci of a conic which consists of a pair of *unlimited* lines coincident therewith. But if we exclude the foci of the doubled line through a and a', regarded as such an exceptional conic, we may disregard the indeterminate constituent circle ($\beta\gamma$) of (Σ) as well as the constituent line $\beta\gamma$ of ($\Sigma_{,}$), and say that *the foci of a system of conics having double contact at* a *and* a' *with a circle* (O), *lie on the circular cubic consisting of the diameter of* (O) *which bisects the chord* aa', *and of the circle whose diameter is the intercept between the pole of* aa', *and the centre of* (O).

In conclusion I may remark that we should have arrived directly at this result by the method of Chasles, for the characteristics of the present system are both equal to unity. (See my Solution of Question 1717, *Reprint*, Vol. IV., p. 19.)

1853. (Proposed by J. WILSON.)—Find two series of integral cubes such that every term in the first may be the sum, and every term in the second the difference, of two integral squares. Also find two series of integral squares, such that every term in the first may be the sum, and every term in the second the difference, of two integral cubes.

Solution by SAMUEL BILLS.

First; to find cube numbers which shall be the sum or difference of two squares, assume the equation $x^3 = p^2x^2 \pm q^2x^2$, then $x = p^2 \pm q^2$, where p and q may be taken at pleasure. This answers the first part of the question.

Again; to find square numbers which shall be the sum or the difference of two cubes, assume the equation $a^2x^2 = p^3x^3 \pm q^3x^3$, then $x = \dfrac{a^2}{p^3 \pm q^3}$, where a, p and q may be taken at pleasure; which satisfies the second part of the question. It is evident that an indefinite number of answers may be found in each case. To obtain integers, a^2 must be divisible by $p^3 \pm q^3$.

2240. (Proposed by the Rev. R. H. WRIGHT, M.A.)—If a conic be described about a triangle ABC, and tangents at (B, C), (C, A), (A, B) meet respectively in G, H, K; then, if D, E, F be any three points in BC, CA, AB such that AD, BE, CF are concurrent, the three lines GD, HE, KF will also be concurrent.

I. *Solution by* S. WATSON; W. CHADWICK; W. H. LAVERTY; J. DALE; *the* PROPOSER; *and others.*

Let the equation of the conic be $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$, then the equations of the tangents at A, B, C are respectively

$$\frac{\beta}{m} + \frac{\gamma}{n} = 0,\quad \frac{\gamma}{n} + \frac{\alpha}{l} = 0,\quad \frac{\alpha}{l} + \frac{\beta}{m} = 0 \ldots\ldots (1, 2, 3);$$

also the equations of AD, BE, CF are all comprehended in

$$p\alpha = q\beta = r\gamma \ldots\ldots\ldots\ldots\ldots (4).$$

From the above equations, those of the three lines GD, HE, KF are easily found to be all comprehended in

$$rp\,(l\gamma + n\alpha) = pq\,(m\alpha + l\beta) = qr\,(n\beta + m\gamma) \ldots\ldots (5);$$

hence GD, HE, KF meet in the point determined by (5).

II. *Solution by* ARCHER STANLEY; W. CHADWICK; *and others.*

It is manifest that the equations

$$\frac{\sin DGB}{\sin DGC} = \frac{DB}{DC} \cdot \frac{BG}{CG},\quad \frac{\sin EHC}{\sin EHA} = \frac{EC}{EA} \cdot \frac{CH}{AH},\quad \frac{\sin FKA}{\sin FKB} = \frac{FA}{FB} \cdot \frac{AK}{BK}$$

are true *numerically*, wherever D, E and F may be situated on BC, CA and AB.

But A, B, C being the points of contact of a conic inscribed in GHK, it

follows from a well known modification of BRIANCHON's theorem that GA, HB and KC are concurrent, and hence, by CEVA's theorem, that unity is the numerical value of the ratio $\frac{BG \cdot CH \cdot AK}{CG \cdot AH \cdot BK}$.

The multiplication of the preceding equations therefore gives

$$\frac{\sin DGB \cdot \sin EHC \cdot \sin FKA}{\sin DGC \cdot \sin EHA \cdot \sin FKB} = \frac{DB \cdot EC \cdot FA}{DC \cdot EA \cdot FB},$$

which is obviously true in sign as well as in magnitude. By CEVA's theorem and its converse therefore we conclude that *whenever* AD, BE, *and* CF *are concurrent,* GD, HE *and* KF *are so likewise.* Moreover by the theorem of MENELAUS we conclude that *whenever* D, E *and* F *are collinear,* GD, HE *and* KF *intersect the sides of* GHK *in three collinear points.*

TO FIND THE FORM OF ALL POSSIBLE INTEGRAL SOLUTIONS OF $a^x \pm 1 \searrow x$; WHERE THE SYMBOL $\searrow$ DENOTES "DIVISIBLE BY."

BY MORGAN JENKINS, B.A.

Let $N \searrow p\,[h^q]$ signify that N is divisible by h^q, but by no higher power of h than that which consists of the product of h^q into the highest power of h contained in p (h being prime). Also let $N \searrow p\,[u]$ denote a corresponding relation for every prime factor of u. In both the proposed cases x must of course be prime to "a."

With respect to $a^x - 1 \searrow x$, we have

(1.) x cannot be prime to $a-1$. For let h be the least prime factor of x, and α the least integer consistent with $a^\alpha - 1 \searrow h$, then $\alpha = (h-1)$ or some measure of $(h-1)$. Therefore, if $h = 2$, $\alpha = 1$. But, if $h > 2$, α cannot be >1; for otherwise, since $a^x - 1 \searrow h$, x would contain $\alpha\,(>1)$ a measure of $(h-1)$, which is contrary to the hypothesis that h is the least prime factor of x.

Therefore $\alpha = 1$, and h is a measure of, and therefore x is not prime to, $a-1$.

COR.—$2^x - 1 \searrow x$ is impossible, and $3^x - 1 \searrow x$ is impossible, unless x is even (excluding $x = 1$ in both cases).

(2.) If $a^{py} - 1 \searrow y$, y can not be prime to $a^p - 1$. This follows from (1), since $a^{py} - 1 = (a^p)^y - 1$.

(3.) If p be any integer and h a prime number, such that

$$a^p - 1 \searrow h, \text{ say } \searrow [h^q] \text{ then } a^{ph^r} - 1 \searrow [h^{q+r}] \searrow (a^p - 1)\,[h^r].$$

For, let $a^p = 1 + mh^q + \text{multiple of } h^{q+1}$; where m is prime to h.

$$a^{ph} = (1 + mh^q + \text{multiple } h^{q+1})^h = 1 + mh^{q+1} + \text{multiple } h^{q+2},$$

therefore $\quad a^{ph} - 1 \searrow [h^{q+1}]$, therefore $a^{ph^2} - 1 \searrow [h^{q+2}]$,

and by induction $\quad a^{ph^r} - 1 \searrow [h^{q+r}] \searrow (a^p - 1)\,[h^r]$.

(4.) These considerations are sufficient to furnish all the solutions of $a^x - 1 \searrow x$. By (1) x cannot be prime to $a-1$. By (3), x may contain any powers of any one or more of the prime factors of $(a-1)$, that is, x may be any measure of any power of $(a-1)$, for if $h, k, \ldots$ be any or all of the prime factors of $(a-1)$ and $x = h^r k^s \ldots$

$$a^x - 1 \searrow (a-1)\,[h^r], \quad \text{and also} \searrow (a-1)\,[k^s] \text{ \&c.}$$

therefore $\searrow (a-1)\,[h^r]\,[k^s]\ldots$, therefore $\searrow (a-1)\,[x]$.

Again, x may contain other factors prime to $(a-1)$. For, let $x = yu$, where $u_{,}$ is a measure of a power of $(a-1)$: then, since $a^{yu_{,}} - 1 \searrow yu, \searrow y$,

by (2), y can not be prime to $a^{u_{,}} - 1$;

by (3), y may be any measure of any power of $(a^{u_{,}} - 1)$,

and $a^{yu_{,}} - 1 \searrow (a^{u_{,}} - 1)\,[y] \searrow (a-1)\,[yu_{,}]$.

Though y might contain some factors of $(a-1)$, yet these may be excluded without loss of generality. For, if y contain a factor u' of the same form as $u_{,}$; say, $y = y'u'$; then y' measures y, andtherefore some power of $(a^{u_{,}} - 1)$, and therefore some power of $(a^{u'u_{,}} - 1)$, and therefore y' might be obtained by beginning with $u'u_{,}$, which is of the same form as $u_{,}$, instead of $u_{,}$; $\therefore$ y may be considered as prime to $(a-1)$. These factors may be said to be of the second order of formation; and we may proceed to find factors of superior orders of formation *ad infinitum*. But, by (2), x cannot contain any factor not formed in the way just pointed out.

The factors of any order of formation may, as has been shown for a particular case, be taken, without loss of generality, so as to be prime to the factors of all preceding orders.

Let U_r denote $u_1 . u_2 \ldots\ldots\ldots\ldots u_r$; then

The complete integral solution of $a^x - 1 \searrow x$ *is*

$x = U_n \equiv u_1 . u_2 \ldots\ldots u_n$; *where*

u_1 *is any measure of any power of* $(a-1)$.

u_2 *is any measure, prime to* $(a-1)$, *of any power of* $(a^{u_{,}} - 1)$.

. . . .

u_n *is any measure, prime to* $(a^{U_{n-2}} - 1)$, *of any power of* $(a^{U_{n-1}} - 1)$.

We also have

$$a^{U_n} - 1 \searrow (a^{U_{n-1}} - 1)\,[u_n],$$

which expression gives the highest powers of factors of the nth order of formation;

$$a^{U_{n-1}} - 1 \searrow (a^{U_{n-2}} - 1)\,[u_{n-1}];$$

and $a^{U_n} - 1$ cannot contain a higher power of any of the prime factors of u_{n-1} than $(a^{U_{n-1}} - 1)$, because $U_n = u_n U_{n-1}$, and u_n is by supposition prime to U_{n-1};

therefore $a^{U_n} - 1 \searrow (a^{U_{n-2}} - 1)\,[u_{n-1}]$,

and so on $\searrow (a-1)\,[u_{,}]$ and also $\searrow U_n$.

If "a" be *odd* $(a-1)$ is even, and therefore $u_{,}$ may be even; therefore x may (if $u_{,}$ be even) contain any measure of any power of (a^2-1); therefore x may contain any measure of any power of $(a+1)$ provided x be even.

In this case, since 2 is contained in both $(a-1)$ and $(a+1)$ we must, in *calculating* the highest powers of the factors of $a \pm 1$, consider 2 as a factor

of the first order only, or more shortly proceed thus. Let $x = 2x'$,

$$a \;\; -1 = (a^2)^{x'} - 1 \searrow (a^2-1)\;[x'] \searrow \frac{a^2-1}{2}\lfloor x].$$

(5.) The solutions of $a^x+1 \searrow x$ may be deduced (with a few modifications) from the previous results. *But x cannot contain 2^2 or any higher power of 2.* For, in this case, a^x+1 is even, and therefore a odd. Let $x = 2x'$. Then $a^{x'}$ is odd and $a^x+1 = (a^{x'})^2+1$ is of the form $(4m^2+4m+1)+1$.

Therefore $\quad a^x+1 \searrow \lfloor 2]$ if a be odd, and a^x+1 is odd if a be even.

(α.) Let x be odd, then in (3) first changing -1 into $+1$, and supposing h to be odd or any prime >2, we have

$$a^{ph^r}+1 \searrow (a^p+1)\,[h^r] \text{ if } a^p+1 \searrow h.$$

Since $a^x+1 \searrow x$ and x is odd, a^x-1 is prime to x, therefore x is prime to $a-1$.

But $a^{2x}-1 = (a^2)^x-1 \searrow x$; therefore x is not prime to (a^2-1) and x is prime to $(a-1)$, therefore x is not prime to $a+1$. In like manner if x be odd and $a^{px}+1 \searrow x$, x cannot be prime to a^p+1.

Hence solutions of $a^x+1 \searrow x$ can be found, similar to those of $a^x-1 \searrow x$, by taking odd factors formed from $(a+1)$ instead of factors formed from $(a-1)$.

(β.) Let x be even $= 2x'$ where x' is odd. Then $a^x+1 = (a^2)^{x'}+1$; therefore x' must not be prime to a^2+1, that is, must be prime to a^2-1; for any odd prime, which measures a^2-1 would also measure $a^{2x'}-1$ or a^x-1, and would therefore be prime to a^x+1.

Hence, when a is odd, beside the solutions indicated in the previous article (5a) we may obtain a different set of solutions consisting of the product of 2 into odd factors formed from a^2+1.

A few illustrations will show the application of these solutions.

$$11^{(3221^p\times 3^q)\cdot(2^2\times 5)}-1 \searrow (11^{2^2\times 5}-1)\,[3221^p]\,[3^q] \searrow [3221^{p+1}]\,[3^{q+1}],$$

3221 being a prime factor of 11^5-1, and 3 of 11^2-1, and therefore each of these prime factors of $11^{2^2\times 5}-1$,

$$11^{(3221^p\times 3^q)\,(2^2\times 5)}-1 \searrow (11-1)\,[2^2]\,[5] \searrow [2^3]\,[5^2],$$

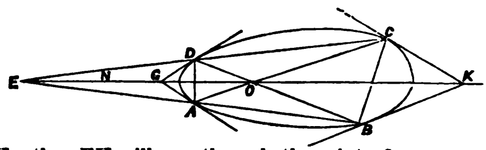
$$5^{7^s\times 3}+1 \searrow (5+1)\,[3] \searrow [3^2] \searrow (5^3+1)\,[7^s] \searrow \lfloor 7^{s+1}],$$

$$5^{13^s\times 2}+1 \searrow (5^2+1)\,[13^s] \searrow [13^{s+1}] \searrow [2].$$

2002. (Proposed by the Rev. R. H. WRIGHT, M.A.)—If a quadrilateral be inscribed in a conic, and either pair of opposite sides BA, CD, be produced to meet in E; then the line joining the point E with G, the intersection of the tangents at A and D, will pass through the intersections of the diagonals of the quadrilateral.

Solution by T. T. WILKINSON, F.R.A.S.; W. CHADWICK; A. COHEN, B.A.; H. TOMLINSON; J. DALE; W. H. LAVERTY; *and many others.*

This question is very neatly solved in Articles 35 and 36 of MACLAURIN'S *General Properties of Geometrical Lines;* and his reasoning may be adapted to the preceding enunciation as follows :—



Let the straight lines AB and CD inscribed in a conic meet in the point E; let straight lines touching the conic in the points B and C meet each other in K: then EK will pass through the point of concourse G of right lines which touch the conic in the points A and D. For if the line EK does not pass through this point of concourse, let it meet one of the lines in G, and the other in N; then since $\frac{1}{\text{EK}} \mp \frac{1}{\text{EG}} = \frac{1}{\text{EK}} \mp \frac{1}{\text{EN}}$, we have EG = EN, and the points G and N coincide, contrary to the hypothesis. For similar reasons it appears that AC and DB meet each other in the point O of the same line EK, and therefore the four points E, G, O, K, are in the same straight line. (MACLAURIN'S *Algebra*, pp. 467, 468.) Dr. SIMSON deduces the same theorem from Prop. 49, Book V., of his *Conic Sections;* but there is nothing novel in his mode of solution. M. CHASLES applies his system of Anharmonic Ratios to the same inquiry as follows :— Premising the property that if round two fixed points on a conic two straight lines revolve so as to intersect on the curve; these lines taken in their successive positions will form two homographic pencils; he finds that D and A are two such points; and since AG, DA; AB, DB; AC, DC; AD, DG; intersect on the curve, we have (A . GBCD) = (D . GCBA), and since the two rays AD and DA coincide, the remaining rays intersect in the points E, G, O, which range in the same straight line. Similar considerations will prove that tangents from B and C will intersect in a point K on the same line. Hence the property is fully proved.

1926. (Proposed by A. RENSHAW.)—Find general methods of investigating similar series to Euler's and Machin's used for the calculation of π, and prove the relation

$$\frac{\pi}{4} = \left\{\frac{1}{\pi} - \frac{1}{3\pi^3} + \frac{1}{5\pi^5} - \&c.\right\} + \left\{\frac{\pi-1}{\pi+1} - \frac{(\pi-1)^3}{3(\pi+1)^3} + \&c.\right\}.$$

Solution by the REV. J. L. KITCHIN, M.A.; W. H. LAVERTY; S. W. BROMFIELD; *the* PROPOSER; *and others.*

We have $\tan^{-1}(x_1^{-1}) + \tan^{-1}(x_2^{-1}) + \ldots. + \tan^{-1}(x)_n^{-1}$

$$= \tan^{-1}\left\{\frac{x_1^{-1} + x_2^{-1} + \ldots. + x_n^{-1} - x_1^{-1}x_2^{-1}x_3^{-1} - \&c.\ldots.}{1 - x_1^{-1}x_2^{-1} - x_1^{-1}x_3^{-1} - \&c.}\right\}$$

For Euler's or Machin's series, and for series similar to them, we must have the second side put equal to $\frac{1}{4}\pi$;

$$\text{therefore}\quad \frac{x_1^{-1}+x_2^{-1}+\ldots.+x_n^{-1}-x_1^{-1}x_2^{-1}x_3^{-1}-\&c.\ldots.}{1-x_1^{-1}x_2^{-1}-x_1^{-1}x_3^{-1}-\&c.\ldots.}=1,$$

$$\text{therefore}\quad x_2x_3\ldots.x_n+x_1x_3\ldots.x_n+\ldots.+x_1\ldots.x_n\ldots.-\&c.$$
$$=x_1x_2\ldots.x_n-x_3x_4\ldots.x_n-\&c.\ldots.$$

This equation solved for integer values will give any number of series similar to Euler's and Machin's.

$$\text{From}\qquad \tan^{-1}(x_1^{-1})+\tan^{-1}(x_2^{-1})=\tan^{-1}\left\{\frac{x_1^{-1}+x_2^{-1}}{1-x_1^{-1}x_2^{-1}}\right\}=\frac{\pi}{4},$$

we have $x_1+x_2=x_1x_2-1$. Any solution of this equation will give a series for $\frac{1}{4}\pi$. The only solution in positive integers is that which gives Euler's series, viz., $x_1=2,\ \ x_2=3$.

$$\text{Again,}\qquad x_2=\frac{x_1+1}{x_1-1};\ \text{put}\ x_1=\pi,\ \text{then}\ x_2=\frac{\pi+1}{\pi-1};$$

$$\text{therefore}\qquad \frac{\pi}{4}=\tan^{-1}\left(\frac{1}{\pi}\right)+\tan^{-1}\left(\frac{\pi-1}{\pi+1}\right)$$
$$=\left\{\frac{1}{\pi}-\frac{1}{3\pi^3}+\frac{1}{5\pi^5}-\&c.\right\}+\left\{\frac{\pi-1}{\pi+1}-\frac{(\pi-1)^3}{3(\pi+1)^3}+\&c.\right\}.$$

2006. (Proposed by A. RENSHAW.)—Two conics expressed by their general equations touch one another at the origin; find the condition that they should touch each other in one other point.

Solution by W. CHADWICK; W. H. LAVERTY; J. DALE; E. MCCORMICK; S. W. BROMFIELD; H. TOMLINSON; *the* PROPOSER; *and others.*

Let $S\equiv ax^2+2hxy+by^2+T=0$ be the equation to one conic; then the equation to the other is of the form $S'\equiv a'x^2+2h'xy+b'y^2+T=0$, where T is the common tangent at the origin; therefore if S and S′ touch each other in another point, $S-S'=0$ represents their chord of contact and the condition that

$$(a-a')\,x^2+2\,(h-h')\,xy+(b-b')\,y^2=0$$

should represent two *coincident* straight lines is

$$(h-h')^2=(a-a')\,(b-b'),$$

which is the condition required in the question.

1777. (Proposed by Rev. J. BLISSARD.)—Prove that

$$\Delta^r\log^m(1+\Delta)\,0^n=n\,(n-1)\ldots.(n-m+1)\,\Delta^r0^{n-m}.$$

N.B.—From this formula, when r is positive and less than m, by putting $n+m$ for n, we get

$$(n+1)\,(n+2)\ldots.(n+m)\,\Delta^{-r}0^n=\Delta^{m-r}(1-\tfrac{1}{2}\Delta+\tfrac{1}{3}\Delta^2-\&c.)^m0^{n+m}.$$

Solution by E. FITZGERALD.

From Herschel's theorem (*Examples on Finite Differences*, p. 68),

$$F\epsilon^{x} = F(1) + F(1+\Delta).0.\frac{x}{1} + F(1+\Delta)0^2.\frac{x^2}{1.2} + \&c. \ldots\ldots (1),$$

therefore $f(\epsilon^{x}-1) = f(0) + f(\Delta)0.\frac{x}{1} + f(\Delta)0^2\frac{x^2}{1.2} + \&c. \ \ldots\ldots\ldots (2),$

Multiply the members of (2) by x^m; then

$$x^m f(\epsilon^{x}-1) = x^m f(0) + f(\Delta)0.\frac{x^{m+1}}{1} + f(\Delta)0.\frac{x^{m+2}}{1.2}\ \&c. \ldots\ldots\ldots (3).$$

But $x^m f(\epsilon^x - 1) = \left\{\log \epsilon^x\right\}^m f(\epsilon^x - 1)$. Expanding this latter by the formula (1) these results

$$\left\{\log \epsilon^x\right\}^m f(\epsilon^x - 1) = 0 + \left\{\log(1+\Delta)\right\}^m f(\Delta)0.\frac{x}{1} + \&c. \ \ldots\ldots\ldots (4).$$

Equating the coefficients of like powers of x in (3) and (4), we get

$$n(n-1)(n-2)\ldots\ldots(n-m+1)f\Delta 0^{n-m} = \left\{\log(1+\Delta)\right\}^m f(\Delta)0^n \ldots (5).$$

If $f\Delta = \Delta^r$ we have the theorem in question. The theorem (5) is given on p. 310 of De Morgan's splendid work on the Calculus, without demonstration; but this can be easily supplied from the hints there given.

1714. (Proposed by F. D. THOMSON, M.A.)—Investigate the general tangential equation to the foci of a conic, and deduce the general equation to the focus of a parabola and the coordinates of its axis.

Solution by W. S. BURNSIDE, B.A.

The following solution of this problem is contained in Chap. XVIII. of SALMON'S *Conics*.

Let $\Sigma = (A, B, C, F, G, H)\ (\alpha, \beta, \gamma)^2$, $\Omega = \alpha^2 + \beta^2 + \gamma^2 - 2\beta\gamma\cos A - 2\gamma\alpha\cos B - 2\alpha\beta\cos C$; then $\Sigma + k\Omega = 0$ will represent points, where k is determined by the equation $\Theta k^2 + \Delta\Theta_1 k + \Delta^2 = 0$. Eliminating k, $\Theta\Sigma^2 - \Delta\Theta_1\Sigma\Omega + \Delta^2\Omega^2 = 0$ is "the general tangential equation of the foci." In the case of a parabola $\Theta = 0$, and this equation becomes $\Theta_1\Sigma - \Delta\Omega = 0$; whence, if (x_1, y_1, z_1) (x_2, y_2, z_2) be the coordinates of the foci, $x_1x_2 : y_1y_2 : z_1z_2 = \Theta_1 A - \Delta : \Theta_1 B - \Delta : \Theta_1 C - \Delta$, also $x_2 : y_2 : z_2 = A\sin A + H\sin B + G\sin C : H\sin A + B\sin B + F\sin C : G\sin A + F\sin B + C\sin C$, (SALMON'S *Conics*, Art. 293), (x_2, y_2, z_2) being the coordinates of the infinitely distant focus.

The equation of the axis is

$$x(y_1z_2 - y_2z_1) + y(z_1x_2 - z_2x_1) + z(x_1y_2 - x_2y_1) = 0,$$

where $x_1 : y_1 : z_1 = \dfrac{\Theta_1 A - \Delta}{A\sin A + H\sin B + G\sin C} : \dfrac{\Theta_1 B - \Delta}{H\sin A + B\sin B + F\sin C}$

$$: \frac{\Theta_1 C - \Delta}{G\sin A + F\sin B + C\sin C};$$

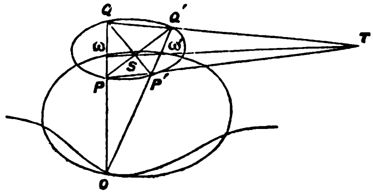
and $x_2 : y_2 : z_2$ are as given above.

I may mention that $\Theta_1\Sigma - \Delta\Omega = 0$ is in general the tangential equation of the envelope of a chord such that the tangents to Σ at its extremities cut at right angles.

A PROPERTY OF A CUBIC AND THE POLAR CONIC OF A POINT UPON IT.

By THOMAS COTTERILL, M.A.

THEOREM.—*If a transversal through a point on a cubic meets the curve again in two points, and the polar conic of the point to the cubic in a third point; then the tangents to the curves at these three points are concurrent.*



For, let lines OPQ, OP′Q′ through O a point on the cubic meet it again in PQ, P′Q′, and the polar conic of O in ω, ω' respectively, so that in the limit PP′, QQ′, $\omega\omega'$ respectively coincide; and let PQ′, QP′ intersect in S, and PP′, QQ′ in T. Then by the harmonic properties of the quadrilinear of which PQ, P′Q′, ST are opposite points, Oω, Oω' are harmonic conjugates to PQ and P′Q′; hence, by the harmonic properties of the polar conic, $\omega\omega'$ must be points upon it. Now, in the limit, when PP′, QQ′ and $\omega\omega'$ respectively coincide, the lines through them become tangents to the curves, which therefore meet in a point.

One of the consequences of this simple property is, that a point on the conic and the corresponding intersection of the tangents are inverse points to a self-conjugate triangle of the conic, to which the cubic is its own inverse: so that if $ax^2+by^2+cz^2=0$ is the equation to the conic referred to this triangle, the locus of the intersection of the tangents is of the form

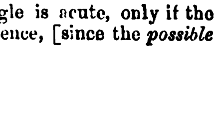
$$\alpha y^2z^2+\beta z^2x^2+\gamma x^2y^2=0.$$

1793. (Proposed by M. W. CROFTON, B.A.)—1. If two sides of a triangle be given, and the third side be taken at random (from among all its possible values), find the probability that the triangle is acute-angled.

2. Two points are taken at random in a given line (l); find the probability of the distance between them exceeding a given length (c).

3. In a plane triangle if arbitrary values be taken for a, b, and C (two sides and the contained angle), and the extreme limit be the same for a and b; find the probability that the triangle is obtuse-angled.

Solution by the PROPOSER; E. FITZGERALD; *and others.*



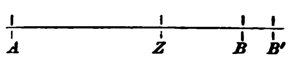
1. Given a, b, two sides of a triangle, the third being taken arbitrarily; to find the chance of the triangle being acute.

Let CB $= a$, the greater of the two given sides; from C as centre draw a circle of radius b; the vertex is on the semi-circle XVZ. Now, the triangle is acute, only if the vertex is on the arc VW, BW being a tangent. Hence, [since the *possible*

values of c range over BX − BZ, and those which give the favourable cases over BV − BW,] it is easily seen that the required chance is

$$p = \frac{\sqrt{(a^2+b^2)} - \sqrt{(a^2-b^2)}}{2b}.$$

If $a = \sqrt{5}$ and $b = 2$, we have $p = \frac{1}{2}$, or it is then an *even* chance that the triangle is acute.



2. Let the given line AB take an increment $dl = BB'$, c remaining unchanged; if F be the measure of the favourable cases, make $BZ = c$, and we have $dF = AZ \,.\, dl = (l-c)\,dl$,

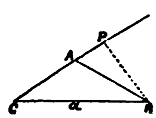
therefore $F = \frac{1}{2}l^2 - cl + C$, C being a function of c.

Now, if $l = c$, $F = 0$; hence $C = \frac{1}{2}c^2$, and $F = \frac{1}{2}(l-c)^2$.

Hence we have $p = \dfrac{F}{\frac{1}{2}l^2} = \left(\dfrac{l-c}{l}\right)^2$.

Otherwise: suppose one point chosen at a distance x from one end of the line, and suppose the distance c set off from this towards each end of the line; then any point chosen between either of these points and the ends will give a favourable case with the point x. The chance of one of these is plainly $\dfrac{x-c}{l}$, and of the other $\dfrac{l-x-c}{l}$. That the *first* should give a real chance, x must lie between c and l; and for the *second* x must lie between 0 and $l-c$. Now the chance of x being taken in any particular position is $\dfrac{dx}{l}$; and consequently the sum of the compound chances gives for the chance required

$$p = \int_0^{l-c} \frac{l-x-c}{l} \cdot \frac{dx}{l} + \int_c^l \frac{x-c}{l} \cdot \frac{dx}{l} = \left(\frac{l-c}{l}\right)^2.$$



3. Let ABC be the triangle; the chance of C being obtuse is $\frac{1}{2}$. To find the chance for A, let k be the extreme limit of value for a and b, draw BP perpendicular to b; then if a and C are supposed fixed, the chance that A is obtuse is clearly $\dfrac{a \cos C}{k}$.

Let now C vary from $\frac{1}{2}\pi$ to 0, a remaining fixed; then

$$p = \frac{a}{k}\int_0^{\frac{1}{2}\pi} \cos C \,.\, \frac{dC}{\pi} = \frac{a}{\pi k}.$$

If a vary now from k to 0, the chance that A is obtuse is

$$p = \int_0^k \frac{a}{\pi k} \cdot \frac{da}{k} = \frac{1}{2\pi}.$$

The same value holds for the angle B, hence the whole probability that the triangle is obtuse-angled is $\dfrac{1}{2} + \dfrac{1}{\pi}$.

1970. (Proposed by Professor CAYLEY.)—Find the conditions in order that the conics
$U = (a, b, c, f, g, h)(x, y, z)^2 = 0$, $U' = (a', b', c', f', g', h')(x, y, z)^2 = 0$,
may have double contact.

Solution by the PROPOSER.

The coefficients of the two conics must be so related that for a properly determined value of θ we shall have identically $U - \theta U' = (\lambda x + \mu y + \nu z)^2$; but when this is so, the inverse coefficients of the quadric function $U - \theta U'$ are each $= 0$; that is, writing

$$(A, B, C, F, G, H) = (bc - f^2, ca - g^2, ab - h^2, gh - af, hf - bg, fg - ch)$$
$$(A', B', C', F', G', H') = (b'c' - f'^2, \ldots, \ldots, g'h' - a'f', \ldots, \ldots)$$
$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}) = (bc' + b'c - 2ff', \ldots, gh' + g'h - af' - a'f, \ldots),$$

then we have the six equations $A - \theta\mathfrak{A} + \theta^2 A' = 0$, &c.

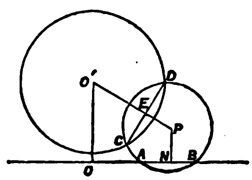
Or, eliminating θ, the required conditions are

$$\left\| \begin{matrix} A, & B, & C, & F, & G, & H \\ A', & B', & C', & F', & G', & H' \\ \mathfrak{A}, & \mathfrak{B}, & \mathfrak{C}, & \mathfrak{F}, & \mathfrak{G}, & \mathfrak{H} \end{matrix} \right\| = 0,$$

equivalent to three relations between the two sets of coefficients.

1972. (Proposed by R. TUCKER, M.A.)—Find the envelope and locus of centres of a system of circles which intercept constant lengths on a fixed line and a fixed circle.

Solution by the PROPOSER.



Let the fixed straight line and the perpendicular on it from the centre (O') of the given circle be the axes of coordinates; and let the variable circle (P) intersect the line in AB ($=2a$) and the circle in CD (chord CD $= 2k$). Then, if $O'O = b$, $O'D = c$, we have

$$a^2 + y^2 = (AP)^2 = k^2 + (PE)^2,$$
$$(O'P)^2 = x^2 + (b-y)^2, \quad (O'E)^2 = c^2 - k^2;$$

hence we have for the locus of P (x, y)

$$\sqrt{\{x^2 + (b-y)^2\}} = \sqrt{(c^2 - k^2)} + \sqrt{(a^2 + y^2 - k^2)},$$

or, $$x^2 - 2by + \lambda^2 = \mu\sqrt{(a^2 + y^2 - k^2)},$$

if $$\lambda^2 = 2k^2 + b^2 - c^2 - a^2 \text{ and } \mu = 2\sqrt{(c^2 - k^2)},$$

that is, $$x^4 + 4b^2y^2 + \lambda^4 - 4bx^2y + 2\lambda^2x^2 - 4b^2\lambda y = \mu^2(a^2 + y^2 - k^2),$$

or, $$x^4 + 2x^2(\lambda^2 - 2by) + 4b^2y^2 + \lambda^4 - 4b\lambda^2 y - \mu^2(a^2 + y^2 - k^2) = 0.$$

If $a = k$, this reduces to the parabolas

$$x^2 - 2by + \lambda^2 = \pm\mu y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)},$$

and if $c = k$ or $\mu = 0$, we have the parabola

$$x^2 - 2by + \lambda^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

For the envelope of the circles we have, in the case (i),

$$X^2 + Y^2 - 2Xx - 2Yy + x^2 = a^2, \qquad x^2 - 2by + \lambda^2 = \pm \mu y;$$

hence $\dfrac{dy}{dx} = -\dfrac{2x}{2b \pm \mu} = \dfrac{x - X}{y}$, and $x = \dfrac{b'X}{b' - 2Y}$, if $b' = 2b \pm \mu$.

Substituting and reducing, we get

$$\left\{ b'(Y^2 - a^2) - 2\lambda^2 Y \right\} (b' - 2Y)^2 = 2b'X^2 Y (b' - 2Y),$$

or,

$$2b'X^2 Y = (b' - 2Y)\left\{ b'Y^2 - 2\lambda^2 Y - a^2 b' \right\},$$

hence the envelope on this supposition is a cubic curve.

1990. (Proposed by Professor SYLVESTER.)—Prove that the locus of one set of foci of all conics passing through four points on a circle is a circular cubic.

Solution by W. S. BURNSIDE, M.A.

Before proceeding with the solution, it may be well to re-state the question in such a form as to give a cue to what will follow.

The locus of the foci of conics passing through four points is in general a curve of the sixth degree; but when the points lie on a circle, the locus resolves itself into two circular cubics, having the four points for foci; also the real foci of the conics lie on one cubic, and the imaginary foci on the other cubic.

MOEBIUS has given the following relation connecting four points, A, B, C, D, on a conic with the focus O:

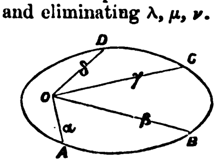
$$\text{OA (BCD)} - \text{OB (CDA)} + \text{OC (DAB)} - \text{OD (ABC)} = 0 \quad\ldots\ldots\ldots \text{(I.)},$$

which is easily seen to be true, by writing the coordinates of the four points in the equation of the conic $\rho = \lambda x + \mu y + \nu$ (origin at the focus), and eliminating λ, μ, ν; and this is the form under which we will consider the equation of the locus of the foci.

FEUERBACH has given the relation connecting four points on a circle with any arbitrary point, viz.

$$\text{OA}^2\text{(BCD)} - \text{OB}^2\text{(CDA)} + \text{OC}^2\text{(DAB)} - \text{OD}^2\text{(ABC)} = 0 \ldots\ldots \text{(II.)},$$

which also follows by substituting the coordinates of the four points in the equation of the circle $\rho^2 = \lambda x + \mu y + \nu$ (origin at O), and eliminating λ, μ, ν.



We now write the last two relations in the shorter forms

$$l\alpha + m\beta + n\gamma + r\delta = 0 \quad\ldots\ldots\ldots\ldots (1),$$

$$l\alpha^2 + m\beta^2 + n\gamma^2 + r\delta^2 = 0 \quad\ldots\ldots\ldots\ldots (2),$$

with

$$l + m + n + r \equiv 0 \quad\ldots\ldots\ldots\ldots (3).$$

Eliminating δ, we have

$$r(l\alpha^2 + m\beta^2 + n\gamma^2) + (l\alpha + m\beta + n\gamma)^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Forming now the discriminant of the left-hand side of the equation, it is found to be $lmnr^2(l+m+n+r)$, which vanishes in virtue of (3); whence the first part of the theorem is true, as (4) is resolvable into

$$(l_1\alpha+m_1\beta+n_1\gamma)(l_2\alpha+m_2\beta+n_2\gamma)=0.$$

Again, these factors are actually

$$lm(\alpha-\beta)+ln(\alpha-\gamma)\pm(lmnr)^{\frac{1}{2}}(\beta-\gamma),$$

showing that $l_1+m_1+n_1\equiv 0$, and $l_2+m_2+n_2\equiv 0$, or that the two constituents of the locus are circular cubics having the points A, B, C for foci.

From $l_1\alpha+m_1\beta+n_1\gamma=0$ combined with $l_1+m_1+n_1\equiv 0$, we see that, if the point (α, β, γ) lies on this cubic, so does the point $(\alpha+h, \beta+h, \gamma+h)$; and these points may be considered as conjugate foci.

As we might equally well have retained the vectors (β, γ, δ) only in the equations of the cubics, it appears that all four points are foci of each of them.

2230. (Proposed by M. HERMITE.)—Soit $F(x)$ un polynome qui reste positif pour toutes les valeurs réelles de la variable; il en sera de même du polynome suivant:

$$\Phi(x)=F(x)+aF'(x)+a^2F''(x)+a^3F'''(x)+\&c.$$

quel que soit la constante a.

Et si le polynome $F(x)$ est quelconque, la plus grande racine de l'équation $\Phi(x)=0$ sera inférieure à la plus grande des racines de $F(x)=0$, si la constante a est positive.

Plus généralement, soit $\Theta(x)=1+\alpha x+\beta x^2+\&c.=0$ une equation dont toutes les racines *sont réelles et positives;* si l'on fait

$$\frac{1}{\Theta(x)}=1+ax+bx^2+cx^3+\&c.,$$

la plus grande racine réelle de

$$\Phi(x)=F(x)+a\,F'(x)+b\,F''(x)+c\,F'''(x)+\&c.$$

sera audessous de la plus grande racine réelle de $F(x)=0$, et si le polynome $F(x)$ est positif quel que soit x, il en sera de même de $\Theta(x)$. Seulement alors il suffit que toutes les racines de $\Theta(x)=0$ soient réelles, sans être toutes positives.

Solution by THOMAS SAVAGE, M.A.

1. It is evident that $\Phi(x)-a\dfrac{d}{dx}\Phi(x)=F(x)$, or multiplying both sides by

$\dfrac{1}{a}\epsilon^{-\frac{x}{a}}$, we have $\dfrac{d}{dx}\left\{-\epsilon^{-\frac{x}{a}}\Phi(x)\right\}=\dfrac{1}{a}\epsilon^{-\frac{x}{a}}F(x).$

Now if a be positive, the right-hand side of this equation is always positive. Hence the function $-\epsilon^{-\frac{x}{a}}\Phi(x)$ must constantly increase as x increases from $-\infty$ to $+\infty$, and therefore if it is negative when $x=+\infty$ it is always

negative. But it is negative when $x = +\infty$; for the terms which involve the highest powers of x in $F(x)$ and $\Phi(x)$ are the same, and when x is very large the sign of these terms determines the signs of $F(x)$ and $\Phi(x)$; hence when x is large $\Phi(x)$ is positive, and $-\epsilon^{-\frac{x}{a}}\Phi(x)$ is negative. And since $-\epsilon^{-\frac{x}{a}}\Phi(x)$ is always negative, $\Phi(x)$ is always positive.

Again, let a be negative. It is easily seen from (1) that the function $-\epsilon^{-\frac{x}{a}}\Phi(x)$ continually diminishes as x increases from $-\infty$ to $+\infty$, and as before, when $x = -\infty$, $F(x)$ and $\Phi(x)$ have the same sign, or $-\epsilon^{-\frac{x}{a}}\Phi(x)$ is negative; it is therefore always negative and $\Phi(x)$ is therefore always positive.

2. Suppose now that $F(x)$ is not always positive, but that its greatest root is k. Then for values of x greater than k, $F(x)$ is always positive; and if a be positive, it may be shown as before, that as x increases from k to $+\infty$, the function $-\epsilon^{-\frac{x}{a}}\Phi(x)$ also continually increases; and when x is very large, this function is negative; it is therefore negative for all values of x greater than k, or $\Phi(x)$ is positive for all such values, and the greatest root of $\Phi(x)$ is less than k.

It is also easily seen that if a be negative, *and the degree of* $F(x)$ *be even*, the numerically greatest negative root of $\Phi(x)$ is (numerically) less than the greatest negative root of $F(x)$.

3. Again, let $\frac{1}{p}, \frac{1}{q}, \frac{1}{r} \ldots$ be the roots of the equation $\Theta(x) = 0$. Then we have (separating the symbols of operation)

$$\Phi(x) = \frac{F(x)}{\Theta\left(\frac{d}{dx}\right)} = \frac{F(x)}{\left(1-p\frac{d}{dx}\right)\left(1-q\frac{d}{dx}\right)\left(1-r\frac{d}{dx}\right)\ldots},$$

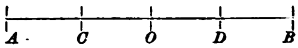
and if $p, q, r \ldots$ be all positive, and k the greatest root of $F(x)$, it has been shown that the greatest root of $\frac{F(x)}{1-p\frac{d}{dx}}$ is less than k. Let it be equal to k'.

Then the greatest root of $\frac{F(x)}{\left(1-p\frac{d}{dx}\right)\left(1-q\frac{d}{dx}\right)}$ is less than k'. Proceeding in this way we see that the greatest root of $\Phi(x)$ is less than k.

If $F(x)$ be always positive, then, whether p, q, r, &c., be positive or negative, $\frac{F(x)}{1-p\frac{d}{dx}}$ will be always positive; therefore also $\frac{F(x)}{\left(1-p\frac{d}{dx}\right)\left(1-q\frac{d}{dx}\right)}$ will be always positive; and thus it may be proved that $\frac{F(x)}{\Theta\left(\frac{d}{dx}\right)}$, or $\Phi(x)$, is always positive.

2257. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—A line, fixed in length and position, is cut at two variable points into three segments the sum of whose squares is constant; required the locus of the vertex of the equilateral triangle described on the middle segment as base.

Solution by S. WATSON; W. H. LAVERTY; W. CHADWICK; E. MCCORMICK; *and others.*



Let O be the middle point of the given line AB, and C, D the dividing points. Take OB and a perpendicular to it at O for axes, and put AB $= a$, AC $= z$, CD $= z'$; then

$$z^2 + z'^2 + (a - z - z')^2 = c^2 \text{ (a constant)} \quad \ldots\ldots\ldots\ldots\ldots (1).$$

Also the coordinates of the vertex of the equilateral triangle on CD are

$$x = z + \tfrac{1}{2}z' - \tfrac{1}{2}a, \text{ and } y = \tfrac{1}{2}z'\sqrt{3} \quad \ldots\ldots\ldots\ldots\ldots (2);$$

hence eliminating z, z' from (1) and (2), the result, viz.

$$x^2 + y^2 - \tfrac{1}{3}\sqrt{3} \,.\, ay = \tfrac{1}{2}c^2 - \tfrac{1}{4}a^2,$$

is the equation of the required locus, and represents a circle.

[When $c = a$, the circle passes through A and B; when $c^2 = \frac{1}{2}a^2$ the circle touches AB; when $c^2 = \frac{1}{3}a^2$, the circle becomes a point; and when $c^2 < \frac{1}{3}a^2$, the circle becomes imaginary, showing that the line cannot be divided so that the squares on the three segments shall be less than one-third of the square on the whole line, as is indeed obvious from other considerations.]

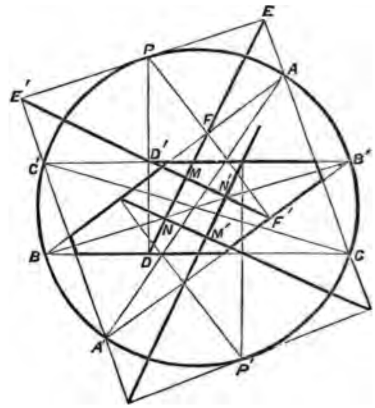
1495. (Proposed by HUGH GODFRAY, M.A.)—Show that $\frac{1}{6}n(n-1)(n-2)$ points can always be so arranged in a plane that they shall be situated by eights in $\frac{1}{24}n(n-1)(n-2)(n-3)$ circles.

Solution by the PROPOSER.

Let us consider n points in space. Through any four of these a sphere may be drawn, and the n points taken in sets of four will give $\frac{1}{24}n(n-1)(n-2)(n-3)$ spheres. Again, through any three of the points a circle may be drawn, and the n points taken three at a time will furnish $\frac{1}{6}n(n-1)(n-2)$ circles; and it is easily seen that each sphere will have four of the circles on its surface. Now, draw a plane so as to cut all the spheres and all the circles. The intersection with each circle will give *two* points on the plane; and the intersection with each sphere will give a circle on the plane, which circle will contain the *eight* points of intersection with the four circles on that sphere. Therefore there will be $\frac{1}{6}n(n-1)(n-2)$ points, situated by *eights* on $\frac{1}{24}n(n-1)(n-2)(n-3)$ circles.

1939. (Proposed by R. TUCKER, M.A.)—Prove that the two feet-perpendicular lines corresponding to any point on the circumscribing circle of a pair of *diametral* triangles intersect at right angles on an ellipse tangential to the six sides. [ABC, A′B′C′ are called *diametral* triangles when AA′, BB′, CC′ intersect in the centre of the common circumscribing circle.]

Solution by JAMES DALE; *and others.*



Drawing the figure, let DEF, D′E′F′, be the feet-perpendiculars of the triangles ABC, A′B′C′, corresponding to the point P; and let 2θ be the angle subtended by AP at the centre of the circle. Then in the triangle D′DM, the angle D′DM = FBP = θ; the angle DD′M = PD′E′ = PC′E′ = PB′A′ = $90^\circ - \theta$; therefore DD′M + D′DM = 90°, and consequently the angle at M is a right angle.

Taking ABC as the triangle of reference, the trilinear coordinates of M are

$$x = \mathrm{DM}\cos\theta = \mathrm{DD'}\cos^2\theta,$$
$$y = \mathrm{EM}\cos\mathrm{MEE'} = \mathrm{EM}\cos(\mathrm{C}-\theta) = \mathrm{EE'}\cos^2(\mathrm{C}-\theta),$$
$$z = \mathrm{FM}\cos\mathrm{PFE} = \mathrm{FM}\cos(\mathrm{B}+\theta) = \mathrm{FF'}\cos^2(\mathrm{B}+\theta);$$

also $\mathrm{DD'} = 2\mathrm{R}\cos\mathrm{A}$, $\mathrm{EE'} = 2\mathrm{R}\cos\mathrm{B}$, $\mathrm{FF'} = 2\mathrm{R}\cos\mathrm{C}$;

therefore $a(x\sec\mathrm{A})^{\frac{1}{2}} + b(y\sec\mathrm{B})^{\frac{1}{2}} + c(z\sec\mathrm{C})^{\frac{1}{2}}$

$$= (2\mathrm{R})^{\frac{1}{2}}\left\{\pm a\cos\theta \pm b\cos(\mathrm{C}-\theta) \pm c\cos(\mathrm{B}+\theta)\right\} = 0.$$

The equation $a(x\sec\mathrm{A})^{\frac{1}{2}} + b(y\sec\mathrm{B})^{\frac{1}{2}} + c(z\sec\mathrm{C})^{\frac{1}{2}} = 0$ represents an ellipse having its centre at the centre of the circumscribing circle, and touching the sides BC, CA, AB, B′C′, C′A′, A′B′ at the points where these lines are intersected by perpendiculars from A′, B′, C′, A, B, C.

If we draw the feet-perpendiculars corresponding to the *diametral* point P′, these, with the two corresponding to P, form a rectangle MNM′N′, whose angles M, M′ lie on the tangential ellipse, while N, N′ lie respectively on the nine-point circles of the triangles ABC, A′B′C′.

2226. (Proposed by H. TOMLINSON.)—If a conic have double contact with two other conics, prove that the chords of intersection of these two conics both pass through the intersection of the chords of contact of the two conics with the first conic.

I. *Solution by* A. COHEN, B.A.; E. MCCORMICK; H. TOMLINSON; W. CHADWICK; *the* PROPOSER; *and others.*

Let $S = 0$ be the equation to the conic. Then the equations to the conics having double contact with the conic $S = 0$ will be of the form $S - k^2\alpha^2 = 0$, $S - k'^2\alpha'^2 = 0$, and by subtracting we find that the chords of intersection of these two conics have for their equations $k\alpha - k'\alpha' = 0$, $k\alpha + k\alpha' = 0$, which therefore both pass through the intersection of $\alpha = 0$, $\alpha' = 0$, or of the chords of contact of the two conics with the first.

II. *Solution by* ARCHER STANLEY.

Let the conic (S) have double contact in A and B with (Σ), and let the latter be touched by the third conic (S′) in A′ and B′. The intersection p of AB and A′B′ will *obviously* have the same polar P relative to all three conics. Hence if a be an intersection of S and S′, and pa intersect P in π, it will intersect both S and S′ again in the harmonic conjugate of a relative to p and π; that is to say, $p\pi$ will be a common chord of S and S′, and in like manner the remaining common chord passes through p.

1521. (Proposed by J. M. WILSON, M.A., F.G.S.)—Show that, in a geometric progression of an odd number of terms, the arithmetic mean of the odd-numbered terms is greater than the arithmetic mean of the even-numbered terms, if the common ratio be any positive rational quantity not equal to unity.

Solution by C. TAYLOR, M.A.

Although Dr. INGLEBY's solution (*Reprint*, Vol. V., p. 26), leaves nothing to be desired in point of elegance, another inductive proof may perhaps interest those to whom the theorem has "presented great difficulties."

Let F_n denote the fraction $\dfrac{1 + a^2 + a^4 + \ldots + a^{2n}}{a + a^3 + \ldots + a^{2n-1}}$, and D_n its denominator;

then we have to prove that $F_n \equiv \dfrac{1 + aD_n}{D_n} > \dfrac{n+1}{n}$, when $a > < 1$.

Since the *arithmetical mean of any number of positive quantities is greater than the geometrical mean,* (see Todhunter's *Algebra*, Art. 676,)

therefore $$D_n > na^n \text{ and } D_{n+1} > (n+1)\,a^{n+1}.$$

But $$F_n - F_{n+1} = \frac{1}{D_n} - \frac{1}{D_{n+1}} = \frac{a^{2n+1}}{D_n \cdot D_{n+1}} < \frac{1}{n\,(n+1)},$$

therefore $$F_{n+1} > F_n - \frac{1}{n\,(n+1)}, \text{ and, } a \textit{ fortiori},$$

if $$F_n > \frac{n+1}{n}, \text{ we have } F_{n+1} > \frac{n+1}{n} - \frac{1}{n\,(n+1)} > \frac{n+2}{n+1}.$$

Hence inductively, we have, $F_n > \frac{n+1}{n}$.

It may be proved that

$$\frac{n}{n+1} F_n > \frac{n-1}{n} F_{n-1} > \dots\dots > \frac{1}{2}\left(a + \frac{1}{a}\right).$$

2222. (Proposed by S. BILLS.)—Prove that if $2x+1$ be any *prime* number, and if a and b be any two unequal whole numbers, not multiples of $2x+1$, then either $a^x + b^x$ or $a^x - b^x$ will be exactly divisible by $2x+1$.

Solution by the REV. J. BLISSARD; T. COTTERILL; W. CHADWICK; J. DALE; E. MCCORMICK; *the* PROPOSER; *and others.*

By FERMAT'S Theorem, $\frac{a^{2x}-1}{2x+1}$ and $\frac{b^{2x}-1}{2x+1}$ are integral. Hence one of the expressions $\frac{a^x-1}{2x+1}$, $\frac{a^x+1}{2x+1}$ must be integral, as also one of the expressions $\frac{b^x-1}{2x+1}$, $\frac{b^x+1}{2x+1}$; and whichever of the first pair be integral and whichever of the second pair, by adding or subtracting as the case may require, either $\frac{a^x-b^x}{2x+1}$, or $\frac{a^x+b^x}{2x+1}$ must be integral. Thus, if $\frac{a^x-1}{2x+1}$ and $\frac{b^x+1}{2x+1}$ are integers, then by adding, $\frac{a^x+b^x}{2x+1}$ must be an integer. For example, if $x = 8$, then either $a^8 + b^8$ or $a^8 - b^8$ is exactly divisible by 17, &c.

1904. (Proposed by J. GRIFFITHS, M.A.)—Let G′ denote the inverse of the centre of gravity G of any triangle ABC; H the equilateral hyperbola which passes through the points A, B, C, G; E the ellipse which touches the sides of the triangle at the points where they are intersected by the lines AG′, BG′, CG′; show that the nine-point circle of the triangle touches the common tangents of the two curves H and E.

I. *Solution by* W. H. LAVERTY.

Let ABC be the triangle of reference. Now, we know that if

$$\Sigma Ax^2 + 2\Sigma Dyz = (A, B, C, D, E, F)(x, y, z)^2 = 0$$

be the equation to a conic, and if A′, B′, C′, D′, E′, F′ are the coefficients of

A, B, C, D, E, F found by expanding $\begin{vmatrix} A & F & E \\ F & B & D \\ E & D & C \end{vmatrix}$; we have $(A', B', C', D', E', F')(l, m, n)^2 = 0$ for the condition that $lx + my + nz = 0$ may touch the conic.

Now we have for G and G′ respectively

$$\frac{x}{a^{-1}} = \frac{y}{b^{-1}} = \frac{z}{c^{-1}}; \text{ and } \frac{x}{a} = \frac{y}{b} = \frac{z}{c};$$

and the equations to H and E are respectively

$$yz \sin(B-C) + zx \sin(C-A) + xy \sin(A-B) = 0, \text{ and } \Sigma \frac{x^2}{a^2} - 2\Sigma \frac{yz}{bc} = 0;$$

and therefore the conditions that $lx + my + nz = 0$ should be a tangent to H and E are

$$\Sigma \left(\frac{(b^2-c^2)^2}{a^2} l^2\right) - 2\Sigma \left(\frac{(c^2-a^2)(a^2-b^2)}{bc} mn\right) = 0,$$

and $$bc \, . \, mn + ca \, . \, nl + ab \, . \, lm = 0.$$

Dividing the first of these by 4 and subtracting it from the other, we have

$$\Sigma \frac{(b^2-c^2)^2}{4a^2} l^2 - \Sigma \left(\frac{bc}{2} + a^2 . \cos A\right) mn = 0,$$

which will be found to be the condition that $lx + my + nz = 0$ should touch the nine-point circle $\Sigma a \cos A \, . \, x^2 - \Sigma a \, . \, yz = 0$.

II. *Solution by* James Dale.

The equations in trilinear coordinates of the hyperbola, ellipse, and circle are as follows:—

$$\frac{b^2-c^2}{a} yz + \frac{c^2-a^2}{b} zx + \frac{a^2-b^2}{c} xy = 0 \quad \ldots\ldots\ldots\ldots\ldots (H),$$

$$\sqrt{\left(\frac{x}{a}\right)} + \sqrt{\left(\frac{y}{b}\right)} + \sqrt{\left(\frac{z}{c}\right)} = 0 \quad \ldots\ldots\ldots\ldots\ldots (E),$$

$$a \cos A \, . \, x^2 + b \cos B \, . \, y^2 + c \cos C \, . \, z^2 - (ayz + bzx + cxy) = 0 \ldots. (C).$$

The conditions that the line $lx + my + nz = 0$ should touch (H), (E), (C) are, respectively,

$$\frac{(b^2-c^2)^2}{a^2} l^2 + \frac{(c^2-a^2)^2}{b^2} m^2 + \frac{(a^2-b^2)^2}{c^2} n^2$$
$$-2(c^2-a^2)(a^2-b^2)\frac{mn}{bc} - 2(a^2-b^2)(b^2-c^2)\frac{nl}{ca} - 2(b^2-c^2)(a^2-b^2)\frac{lm}{ab} = 0, \quad \ldots\ldots\ldots\ldots (1),$$

$$bc \, . \, mn + ca \, . \, nl + ab \, . \, lm = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$(b^2-c^2)\frac{l^2}{a^2} + (c^2-a^2)\frac{m^2}{b^2} + (a^2-b^2)\frac{n^2}{c^2} - 2(b^2c^2 + c^2a^2 + a^2b^2 - a^4)\frac{mn}{bc}$$
$$-2(b^2c^2 + c^2a^2 + a^2b^2 - b^4)\frac{nl}{ca} - 2(b^2c^2 + c^2a^2 + a^2b^2 - c^4)\frac{lm}{ab} = 0 \quad \ldots. (3).$$

Now $(1) + 4(2) \equiv (3)$; hence the values of l, m, n, which satisfy (1) and (2) simultaneously, also satisfy (3); in other words, (1), (2), (3) are inscribed in the same quadrilateral.

1839. (Proposed by W. S. BURNSIDE, M.A.)—If normals be drawn to a conic at the points P, Q; show that a parabola can be described, touching these two normals, the chord PQ, and the axes of the conic; the diameter conjugate to the chord being the directrix. Also verify the following determination of the common tangents to these curves; through the pole of the chord draw the four normals to the conic, the tangents at their feet are the common tangents required.

I. *Solution by* JAMES DALE.

Let $Ax^2 + By^2 = 1$ be the equation to a central conic, having the principal axes for axes of coordinates; and (x', y'), (x'', y'') the coordinates of any two points P, Q on the curve. Then the directrix of the parabola touching the normals at P and Q, the chord PQ, and the major axis, is the line joining the intersection of the perpendiculars of the triangles formed by the normals and the chord, and by the normals and the major axis.

The perpendiculars from P, Q on the normals at Q, P respectively are

$$y-y' = -\frac{A}{B}\left(\frac{x''}{y''}\right)(x-x'), \quad y-y'' = -\frac{A}{B}\cdot\frac{x'}{y'}(x-x'')\ldots\ldots(1, 2).$$

The intersection of the perpendiculars of the triangle formed by the normals and the major axis is given by the equations

$$y = -\frac{A}{B}\cdot\frac{x''}{y''}\left(x-\frac{B-A}{A}x'\right), \quad y = -\frac{A}{B}\cdot\frac{x'}{y'}\left(x-\frac{B-A}{A}x''\right)\ldots(3, 4);$$

also, $y''(1)-y'(2)$, or $y''(3)-y'(4)$, gives for the equation of the directrix

$$y = -\frac{A}{B}\left(\frac{x'-x''}{y'-y''}\right)x, \quad \text{or } y = \frac{y'+y''}{y'+x''}x\ldots\ldots\ldots\ldots(5, 6).$$

Now, (6) passes through the centre and the middle of the chord, and is therefore the conjugate of the chord; and as the minor axis cuts the major axis at right angles, it follows that the minor axis is also a tangent to the same parabola.

For the second part; let (x''', y''') be any point on the conic, then the condition that the intersection of the perpendiculars of the triangle formed by the tangent at (x''', y''') and the normals at (x', y'), (x'', y'') should lie upon the directrix is readily found to be

$$\frac{A}{B}x'''(x'-x'') + \frac{B}{A}y'''(y'-y'') = (A-B)x'''y'''(x'y''-y'x'');$$

and this is also the condition that the point (x''', y''') should lie upon the hyperbola, which, by its intersection with the conic, determines the feet of the normals drawn through the pole of the chord PQ.

II. *Solution by the* PROPOSER.

1. Let the conic be referred to its axes, and the point (α, β) be the pole of the chord. Now, if $\lambda x + \mu y + \nu = 0$ be the equation of the normal at one of the points whose tangent passes through the point (α, β), we easily find that
$$C^2\lambda\mu + \alpha\mu\nu - \beta\nu\lambda = 0,$$
by comparing with the known equation of the normal.

2. Reciprocally, if $\lambda x+\mu y+\nu=0$ be the equation of the tangent at one of the points whose normal passes through the point (α, β), we find the same tangential equation

$$C^2\lambda\mu+\alpha\mu\nu-\beta\nu\lambda=0.$$

3. This equation is satisfied by $(\lambda=0, \mu=0)$, $(\mu=0, \nu=0)$, $(\lambda=0, \nu=0)$; hence the curve of the second degree represented by this equation touches the line at infinity and the axes of the conic. It is also satisfied by $\lambda=\frac{a}{a^2}$, $\mu=\frac{\beta}{b^2}$, $\nu=-1$; hence this curve touches the chord PQ. So we have proved that a parabola can be described touching the lines described in the question.

4. It remains to show that (α, β) is a point in the directrix, or that the tangents drawn therefrom to the parabola are perpendicular. If $\lambda x+\mu y+\nu=0$ be the equation of one of the tangents, then $\lambda\alpha+\mu\beta+\nu=0$; hence, eliminating ν from the equation $C^2\lambda\mu+\alpha\mu\nu-\beta\nu\lambda=0$, we see that the sum of the coefficients of λ^2 and μ^2 is zero, or the tangents to the parabola from the point (α, β) are at right angles: also the centre is plainly another point on the directrix, since it touches the axes: hence the directrix is determined.

5. Denoting the tangential equations of the conic, the two circular points at infinity, and the pole of the chord, by Σ, Ω, Π, respectively, the equation of the parabola may be written as the Jacobian of these functions, which becomes in the general case [$\alpha x+\beta y+\gamma z=0$ equation of chord, $(a, b, c, f, g, h)(x, y, z)^2=0$ equation of conic] when Δ is divided off,

$$\begin{vmatrix} a & h & g \\ \alpha & \beta & \gamma \\ \lambda & \mu & \nu \end{vmatrix} \Omega_1 + \begin{vmatrix} h & b & f \\ \alpha & \beta & \gamma \\ \lambda & \mu & \nu \end{vmatrix} \Omega_2 + \begin{vmatrix} g & f & c \\ \alpha & \beta & \gamma \\ \lambda & \mu & \nu \end{vmatrix} \Omega_3 = 0, \quad \left(\Omega_1=\frac{d\Omega}{d\lambda}, \text{ \&c.}\right).$$

In conclusion, I should state that the above results may be easily proved by reciprocation.

1809. (Proposed by G. Darboux.)—On circonscrit à un triangle quelconque une courbe du second degré telle que les normales aux trois sommets du triangle passent par un point. On demande de prouver que le lieu de ce point est une courbe à centre du troisième ordre. Déterminer cette courbe.

Solution by J. Dale; *and* A. Cohen, B.A.

Let $lyz+mzx+nxy=0$ be the equation of any circumscribing conic, (x', y', z') the coordinates of the point of intersection of the normals.

Then the tangents at A, B, C are respectively

$$\frac{y}{m}+\frac{z}{n}=0, \frac{z}{n}+\frac{x}{l}=0, \frac{x}{l}+\frac{y}{m}=0 \ldots\ldots(1, 2, 3);$$

and the normals at the same points are

$$\frac{y}{y'}-\frac{z}{z'}=0, \quad \frac{z}{z'}-\frac{x}{x'}=0, \quad \frac{x}{x'}-\frac{y}{y'}=0 \ldots(4, 5, 6).$$

But since (1, 2, 3) are respectively perpendicular to (4, 5, 6), we have

$$m(y'+z'\cos A) = n(y'\cos A + z'),\quad n(z'+x'\cos B) = l(z'\cos B + x'),$$
$$l(x'+y'\cos C) = m(x'\cos C + y');$$

hence eliminating l, m, n, we get

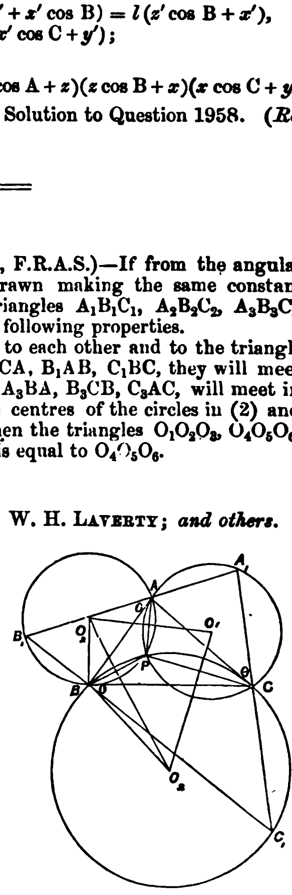
$$(y+z\cos A)(z+x\cos B)(x+y\cos C) = (y\cos A + z)(z\cos B + x)(x\cos C + y),$$

which is the central cubic discussed in the Solution to Question 1958. (*Reprint*, Vol. VI., p. 59.)

1981. (Proposed by T. T. WILKINSON, F.R.A.S.)—If from the angular points of any triangle ABC, lines be drawn making the same constant angles with the adjacent sides, *four* triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, $A_4B_4C_4$, will be formed, which possess the following properties.

(1) The above triangles are all similar to each other and to the triangle ABC; (2) if circles be described about A_1CA, B_1AB, C_1BC, they will meet in a point P; (3) circles described about A_3BA, B_3CB, C_3AC, will meet in another point P_1; (4) if O_1, O_2, O_3 be the centres of the circles in (2) and O_4, O_5, O_6 the centres of those in (3), then the triangles $O_1O_2O_3$, $O_4O_5O_6$, ABC are similar, and the triangle $O_1O_2O_3$ is equal to $O_4O_5O_6$.

Solution by J. DALE; H. TOMLINSON; W. H. LAVERTY; *and others.*



Let B_1C_1, C_1A_1, A_1B_1 make an angle θ towards the same side of BC, CA, AB respectively; then $\angle A_1 = B_1AC - ACA_1 = A$; similarly, $\angle B_1 = B$, and $\angle C_1 = C$; therefore the triangle $A_1B_1C_1$ is similar to ABC. The triangle $A_2B_2C_2$ is formed by making the angle θ towards the *other* side of each of the lines BC, CA, AB; and $A_3B_3C_3$, $A_4B_4C_4$ are formed when B_3C_3, C_3A_3, A_3B_3, and B_4C_4, C_4A_4, A_4B_4 make the same angle θ with CB, AC, BA respectively. The proof that the triangles $A_2B_2C_2$, $A_3B_3C_3$, $A_4B_4C_4$ are similar to ABC is the same as for $A_1B_1C_1$; hence the theorem (1) is proved.

Let circles be drawn round the triangles A_1CA, B_1AB, C_1BC; and let P be the intersection of A_1CA, B_1AB; then, joining AP, BP, CP, the angles CPA, APB, are the supplements of A and B, consequently BPC is the supplement of C or C_1, and therefore P lies on the circle C_1BC, which proves the theorem (2). The proof of (3) is precisely similar.

[The circles A_2CA, B_2AB, C_2BC are identical with A_1CA, B_1AB, C_1BC; and the circles A_4BA, B_4CB, C_4AC are identical with A_3BA, B_3CB, C_3AC.]

Join BO_2, BO_3, and let R = radius of circle ABC; then we have

$$BO_2 = R\frac{c}{b},\quad BO_3 = R\frac{a}{c},\quad \text{and } \angle O_2BO_3 = A+B,$$

$$\therefore\ (O_2O_3)^2 = R^2\left(\frac{a^2}{b^2} + \frac{c}{b^2} + \frac{2a}{b}\cos C\right) = \frac{R^2}{b^2c^2}(b^2c^2 + c^2a^2 + a^2b^2);$$

$$\therefore\ \frac{O_2O_3}{a} = \frac{R}{abc}(b^2c^2 + c^2a^2 + a^2b^2)^{\frac{1}{2}} = \frac{O_3O_1}{b} = \frac{O_1O_2}{c}, \text{ by symmetry.}$$

Similarly if O_4, O_5, O_6 be the centres of A_3BA, B_3CB, C_3AC, we have

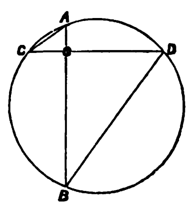
$$CO_5 = R\frac{a}{b},\quad CO_6 = R\frac{b}{c},\ \text{and } \angle O_5CO_6 = A + C,$$

$$\therefore\ \frac{O_5O_6}{a} = \frac{R}{abc}(b^2c^2 + c^2a^2 + a^2b^2)^{\frac{1}{2}} = \frac{O_6O_4}{b} = \frac{O_4O_5}{c};$$

therefore the triangles $O_1O_2O_3$, $O_4O_5O_6$, ABC are similar, and since $O_2O_3 = O_5O_6$, &c., the triangle $O_1O_2O_3$ is equal to $O_4O_5O_6$.

2239. (Proposed by C. M. INGLEBY, LL.D.)—If a draughtman lie on one of the intersections of the board, show that the sum of the arcs bounding the white sectors is always equal to the sum of the arcs bounding the black sectors.

Solution by W. CHADWICK; J. DALE; W. H. LAVERTY; H. TOMLINSON; *the* PROPOSER; *and others.*



Let the circle represent the draughtsman; O an intersection; AD, BC the arcs bounding the white sectors: then, since the angles BAC, DCA are together equal to a right angle, it follows that the sum of the arcs AD, BC is equal to a semicircle, and therefore equal to the sum of the arcs AC, BD.

PROOF OF THREE THEOREMS. BY PROFESSOR EVERETT.

The following proof of three well known theorems is submitted as preferable in some respects, especially in freedom from tentative processes, to those usually given.

THEOREM 1.—$f(a+h)-f(a)$ can be developed in a series of positive integral powers of h, provided that neither $f(x)$ nor any of the derived functions $f'(x), f''(x)$, &c. become infinite or discontinuous between the limits $x = a$ and $x = a + h$.

THEOREM 2.—The form of the development is

$$hf'(a) + \frac{h^2}{\lfloor 2}f''(a) + \frac{h^3}{\lfloor 3}f'''(a) + \ldots.$$

Theorem 3.—The error committed in neglecting all terms of this series after $\frac{h^n}{\lfloor n}f^n(a)$ may be represented by $\frac{h^{n+1}}{\lfloor n+1}f^{n+1}(a+\theta h)$, where θ denotes a positive proper fraction.

We assume, as capable of being independently proved, the three following Lemmas:—

Lemma 1.—If y and z be any variables, the value of $\Sigma(yz)$, taken between such limits that z does not change sign, is equal to $\bar{y}\,\Sigma(z)$, where $\bar{y}$ denotes some quantity intermediate between the algebraically greatest and least values of y.

Lemma 2.—The value of $f(a+h)-f(a)$ is equal to the limit of $\Sigma f'(z)\,\delta z$ from $z=a$ to $z=a+h$, provided that $f'z$ does not become infinite or discontinuous between these limits.

Lemma 3.—The limit of $\Sigma x^n\,\delta x$ from $x=0$ to $x=h$, when n is a positive integer, is $\frac{h^{n+1}}{\lfloor n+1}$.

We are now prepared for the following proof, in which, whenever the sign Σ occurs, the summation indicated is to be taken between the limits $x=0$ and $x=h$, and θ always denotes a positive proper fraction but not always the same fraction.

$$f(a+h)-f(a) = \text{limit of } \Sigma f'(a+x)\,\delta x = f'(a+\theta h)\,\Sigma\,\delta x = f'(a+\theta h)\,h.$$

Hence, writing x for h and f' for f,

$$f'(a+x) = f'(a) + f''(a+\theta x)\,x.$$

Therefore $f(a+h)-f(a)$ being equal to the limit of $\Sigma f'(a+x)\,\delta x$ is equal to the limit of $f'(a)\,\Sigma\,\delta x + f''(a+\theta h)\,\Sigma x\,\delta x$

$$= f'(a)\,h + f''(a+\theta h)\,\frac{h^2}{2}.$$

Hence again, writing x for h and f' for f,

$$f'(a+x) = f'(a) + f''(a)\,x + f'''(a+\theta x)\,\frac{x^2}{2}.$$

Therefore, $f(a+h)-f(a)$ being equal to the limit of $\Sigma f'(a+x)\,\delta x$, is equal to the limit of

$$f'(a)\,\Sigma\,\delta x + f''(a)\,\Sigma x\,\delta x + f'''(a+\theta h)\,\Sigma\,\frac{x^2\,\delta x}{2}$$

$$= f'(a)\,h + f''(a)\,\frac{h^2}{2} + f'''(a+\theta h)\,\frac{h^3}{2.3}.$$

The process here used can obviously be carried to any number of terms; hence Theorems 1, 2, 3 are proved.

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